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On the cohomology of Seifert and graph manifolds

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Abstract

In this survey the cohomology rings $H^*(M^3;\mathbb{Z}_2)$ of orientable Seifert and graph manifolds are described and some proofs are sketched.

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Introduction

The ring structure of the cohomology $H^*(M)$ of a space M is a stronger tool than just the sequence of individual cohomology groups $H^q(M)$, $q = 0, 1, \ldots,$ which, by the universal coefficient theorem, are determined by the homology groups. The calculation of $\cup$-products is in general quite cumbersome and in textbooks of algebraic topology there are only few examples presented. Classical methods are based on chain approximations to the diagonal or the intersection theory of chains and Poincaré duality. Although they are clear in principle and applicable if the space is given by a "nice" complex, there is a great deal of work to calculate $\cup$-products for special cases, compare [3,4].

Recently, in the approach of [1], the cohomology ring of a space M is calculated from those of spaces closely related to subspaces of M. In 5.5 we describe the use of this method for orientable 3-dimensional Seifert and graph manifolds to calculate their cohomology

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rings from those of “simpler” Seifert manifolds, for which the rings were determined in [2,3]. This method has the disadvantage of leading to a solution only by chance.

As an application, for a graph manifold W^3, some kinks are found (which are related to Lorentz metrics on the space–time manifold $W^3 \times \mathbb{R}$, cf. [11]).

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1. On the calculation of ∪-products

In this section we describe the approach from [1] in the updated versions of [5,7,9]. We consider cohomology with coefficients in a fixed commutative ring G with unity. (In fact, in the next sections we are only interested in the case $G = \mathbb{Z}_2$.)

1.1. Notation. Let (X, A) be a pair of CW-complexes and $i : A \hookrightarrow X$ and $j : X \hookrightarrow (X, A)$ be the inclusions of CW-complexes. Then there are the induced homomorphisms

$$\begin{aligned} &i_\bullet : C(A) \to C(X), && i^\bullet : \mathrm{Hom}\big(C(X), G\big) \to \mathrm{Hom}\big(C(A), G\big),\\ &j_\bullet : C(X) \to C(X, A), && j^\bullet : \mathrm{Hom}\big(C(X, A), G\big) \to \mathrm{Hom}\big(C(X), G\big) \end{aligned}$$

on the cellular chain complexes and

$$\begin{aligned} &i^* : H^q(X; G) \to H^q(A; G), && \{\varphi\}_X \mapsto \{\varphi \circ i\}_A,\\ &j^* : H^q(X, A; G) \to H^q(X; G), && \{\varphi\}_{(X,A)} \mapsto \{\varphi\}_X \end{aligned}$$

of the cohomology groups. Here $\{*\}_X$ denotes a cohomology class in X, $\{*\}_A$ a class in A etc. We use the following notation:

$$\begin{aligned} &\varphi|A := \varphi|C_q(A) = \varphi \circ i_\bullet \in \mathrm{Hom}\big(C_q(A), G\big) && \text{for } \varphi \in \mathrm{Hom}\big(C_q(X), G\big) \quad \text{and}\\ &\Phi|A := i^*(\Phi) \in H^q(A; G) && \text{for } \Phi \in H^q(X; G). \end{aligned}$$

1.2. Definition. A cellular cochain $\varphi \in \mathrm{Hom}(C_q(X), G)$ *vanishes on* A if $\varphi(e^q) = 0$ for every q-chain e^q from A; we write $\varphi|A = 0$. A cohomology class $\Phi \in H^q(C(X); G) = H^q(X; G)$ *vanishes on* A if some representative of Φ vanishes on A; we will also write $\Phi|A = 0$.

1.3. Remark. Φ vanishes on A if and only if $i^*\Phi = 0$.

By the exact cohomology sequence

$$\begin{aligned} \cdots \xrightarrow{i^*} H^{q-1}(A; G) \xrightarrow{\delta} H^q(X, A; G) \xrightarrow{j^*} H^q(X; G)\\ \xrightarrow{i^*} H^q(A; G) \xrightarrow{\delta} \cdots \end{aligned}$$

every cocycle of X vanishing on A is also a cocycle of (X, A), and we obtain the following lemma.

1.4. Lemma. *To every cohomology class $\Phi \in H^q(X; G)$ vanishing on A there exists a $\Phi_r \in H^q(X, A; G)$ such that $j^*(\Phi_r) = \Phi$.* □

A consequence of Definition 1.2 is the following proposition from [1].

1.5. Proposition. *Let X be the union of two CW-spaces A and B, $X = A \cup B$, where $A \cap B$ is a subcomplex of A and of B. If $\Phi \in H^p(C(X); G)$ and $\Psi \in H^q(C(X); G)$ then*

(a) $\Phi|A = 0 \implies (\Phi \cup \Psi)|A = 0$;
(b) $\Phi|A = 0,\ \Psi|B = 0 \implies \Phi \cup \Psi = 0$.

Proof. (a) follows from

$$i^*(\Phi \cup \Psi) = i^*(\Phi) \cup i^*(\Psi) = 0 \cup i^*(\Psi) = 0$$

where $i : A \hookrightarrow X$ is the inclusion.

(b) Consider the following diagram:

$$\begin{tikzcd}
H^p(X,A) \arrow[d, "j_A^*"'] & \times & H^q(X,B) \arrow[d, "j_B^*"'] \arrow[r] & H^{p+q}(X, A \cup B) \arrow[d, "\bar{j}^*"'] & (\widetilde{\Phi}, \widetilde{\Psi}) \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
H^p(X) \arrow[d, "i^*"'] & \times & H^q(X) \arrow[d, "i^*"'] \arrow[r] & H^{p+q}(X) & (\Phi, \Psi) \arrow[r, mapsto] \arrow[d, mapsto] & \Phi \cup \Psi \\
H^p(A) & & H^q(B) & & (0,0) &
\end{tikzcd}$$

The elements $\widetilde{\Phi}$ and $\widetilde{\Psi}$ exist because of Lemma 1.4. This diagram is commutative as follows from [12, p. 251 (8)] if one takes $X = Y$, $f = \mathrm{id}_X$, $A_1 = A_2 = \emptyset$, $B_1 = A$, $B_2 = B$ where the left letter is from [12], the right term from this paper. Since (X, A) and (X, B) are CW-pairs they are excisive and the above result can be applied to get

$$0 = \bar{j}^*\big(\widetilde{\Phi} \cup \widetilde{\Psi}\big) = j_A^*\big(\widetilde{\Phi}\big) \cup j_B^*\big(\widetilde{\Psi}\big) = \Phi \cup \Psi.$$

The assertion (b) can also be proved on the chain-cochain level using appropriate chain approximations to the diagonal of A, B, and $A \cup B$. $\square$

To make the following big diagrams a bit simpler we drop the coefficient group G in the rest of this section and write, for instance, $H^p(X)$ instead of $H^p(C(X); G)$.

1.6. Conditions. Let A, A' and B be CW-spaces such that $A \cap B = A' \cap B$ is a subcomplex of A, A', and B. Then $X = A \cup B$ and $X' = A' \cup B$ are CW-spaces. Let

$$\begin{aligned} &i : B \hookrightarrow X, && j : X \hookrightarrow (X, A), \\ &i' : B \hookrightarrow X', && j' : X' \hookrightarrow (X', A') \end{aligned}$$

be the inclusions.

Let exc and exc$'$ be the *excision isomorphisms* in cohomology for the pairs (X, A) and (X', A') when cutting off $X \setminus B$ and $X' \setminus B$, respectively. Then there is the following commutative diagram of cohomology groups.

1.7. Diagram.

$$
\begin{array}{ccccc}
H^p(X') & \times & H^q(X') & \longrightarrow & H^{p+q}(X') \\
\uparrow{\scriptstyle j'^*} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle j'^*} \\
H^p(X',A') & \times & H^q(X') & \longrightarrow & H^{p+q}(X',A') \\
\downarrow{\scriptstyle \mathrm{exc}'\,\cong} & & \downarrow{\scriptstyle i'^*} & & \downarrow{\scriptstyle \mathrm{exc}'\,\cong} \\
H^p(B,B\cap A') & \times & H^q(B) & \longrightarrow & H^{p+q}(B,B\cap A') \\
\| & & \| & & \| \\
H^p(B,B\cap A) & \times & H^q(B) & \longrightarrow & H^{p+q}(B,B\cap A) \\
\uparrow{\scriptstyle \mathrm{exc}\,\cong} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle \mathrm{exc}\,\cong} \\
H^p(X,A) & \times & H^q(X) & \longrightarrow & H^{p+q}(X,A) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle j^*} \\
H^p(X) & \times & H^q(X) & \longrightarrow & H^{p+q}(X)
\end{array}
\qquad
\begin{array}{ccc}
(\Phi',\Psi') & \longmapsto & \Phi'\cup\Psi' \\
\uparrow & & \uparrow \\
(\Phi',\Psi') & \longmapsto & \Phi'\cup\Psi' \\
\downarrow & & \downarrow \\
(\Phi|B,\Psi|B) & \longmapsto & \Phi|B\cup\Psi|B \\
\| & & \| \\
(\Phi|B,\Psi|B) & \longmapsto & \Phi|B\cup\Psi|B \\
\uparrow & & \uparrow \\
(\Phi_r,\Psi) & \longmapsto & \Phi_r\cup\Psi \\
\downarrow & & \downarrow \\
(\Phi,\Psi) & \longmapsto & \Phi\cup\Psi
\end{array}
$$

Consider the following conditions.

(0) $\Phi \in H^p(X)$ and $\Psi \in H^q(X)$ such that Φ vanishes on A.
(1) $\mathrm{exc}'(\ker j'^*) \subset \mathrm{exc}(\ker j^*)$ and j'^* is surjective in dimension $p+q$.
(1′) j'^* is injective in dimension $p+q$.
(2) $i'^*(H^q(X')) = i^*(H^q(X))$.

1.8. Theorem. *Assume that, for given Φ and Ψ, the conditions* (0), (1), (2) *or* (0), (1′), (2) *are fulfilled. Then there are* (*non-unique*) *cohomology classes $\Phi' \in H^p(X')$ and $\Psi' \in H^q(X')$ such that*

$$\Phi \cup \Psi = \left[j^* \circ (\mathrm{exc})^{-1} \circ \mathrm{exc}' \circ \left(j'^*\right)^{-1}\right]\left(\Phi' \cup \Psi'\right) \quad \textit{with}$$
$$\Phi' \in \left[j'^* \circ (\mathrm{exc}')^{-1} \circ \mathrm{exc} \circ (j^*)^{-1}\right](\Phi) \quad \textit{and}$$
$$\Psi' \in \left[\left(i'^*\right)^{-1} \circ i^*\right](\Psi).$$

Proof. By Lemma 1.4, (0) implies that there exists a $\Phi_r \in H^p(X,A)$ such that $\Phi = j^*(\Phi_r)$ and, hence, the $\cup$-product in the second line in the above formula is defined. The equality follows from the commutativity of the diagram and (1) or (1′). □

1.9. Example. A closed orientable surface S_g of genus g admits a cell decomposition $(e^0; t_1^1, t_2^1, q^1, t_3^1, \ldots, t_{2g}^1; e_1^2, e_2^2)$ where the first handle as well as the union of the last $(g-1)$ handles is bounded by q^1: $\partial_2 e_1^2 = q^1$, $\partial_2 e_2^2 = -q^1$. The 1-cells, with the exception of q^1, define a canonical system of generating curves. Let $\tilde{e}^0; \tilde{t}_1^1, \tilde{t}_2^1, \tilde{q}^1, \tilde{t}_3^1, \ldots, \tilde{t}_{2g}^1; \tilde{e}_1^2, \tilde{e}_2^2$ be the corresponding generating cochains. Since all boundaries except those of the e_j^2

vanish the only non-trivial coboundary is $\delta^1\tilde{q}^1 = \tilde{e}_1^2 - \tilde{e}_2^2$. Clearly, the classes $\{\tilde{t}_1^1\}$, $\{\tilde{t}_2^1\}$ vanish on the subcomplex $A = \bar{e}_2^2$ of the last $g-1$ handles; hence, by 1.5,

$$\tilde{t}_j^1 \cup \tilde{t}_k^1 = 0 \quad \text{for } 1 \leqslant j \leqslant 2,\ 3 \leqslant k \leqslant 2g.$$

Let $B = \bar{e}_1^2$ be the first handle and A' a disc: $A' = (e^0; q^1; e_0^2)$, $\partial_2 e_0^2 = -q^1$. *We assume that we know the cohomology ring for the torus* $S_1 = A' \cup B$: the generators are 1; $\{\tilde{t}_1^1\}$, $\{\tilde{t}_2^1\}$; $\Lambda_0 = \{\tilde{e}_1^2 - \tilde{e}_0^2\}$ with the $\cup$-products $\{\tilde{t}_1^1\} \cup \{\tilde{t}_2^1\} = \Lambda_0$, $\{\tilde{t}_j^1\} \cup \{\tilde{t}_j^1\} = 0$ for $j = 1, 2$ as the only $\cup$-products which are not obtained by general properties.

Now we can apply 1.8 to

$$X = S_g = A \cup B, \qquad A \cap B = \partial A = \partial B = \partial A' = A' \cap B, \qquad X' = A' \cup B = S_1,$$

since the conditions (0), (1), and (2) are fulfilled. Hence, we obtain in $H^*(X)$

$$\{\tilde{t}_1^1\} \cup \{\tilde{t}_2^1\} = \Lambda, \qquad \{\tilde{t}_1^1\} \cup \{\tilde{t}_1^1\} = 0, \qquad \{\tilde{t}_2^1\} \cup \{\tilde{t}_2^1\} = 0,$$

where Λ is the generator of $H^2(X) \cong \mathbb{Z}_2$.

By permuting the choice of handles in the above cell decomposition we obtain that in $H^*(S_g)$ the $\cup$-products are determined by

$$\begin{aligned} &\{\tilde{t}_{2i-1}^1\} \cup \{\tilde{t}_{2i}^1\} = \Lambda && \text{for } 1 \leqslant i \leqslant g, \\ &\{\tilde{t}_j^1\} \cup \{\tilde{t}_k^1\} = 0 && \text{for the other cases with } 1 \leqslant j \leqslant k \leqslant 2g. \end{aligned}$$

The diagram from Theorem 1.8 reduces the calculation of the $\cup$-products of X to those of $X' = S_1$ which we assume to be known. Often we know only the cohomology ring of a CW-complex $\widetilde{X}'$ of the same homotopy type as X' and we do not get the relation directly for X, but only for some CW-complex $\widetilde{X}$ of the same homotopy type. Then we can add two more lines to the diagram of Theorem 1.8 and obtain Theorem 1.12.

1.10. Conditions. Let X, $\widetilde{X}$, X', $\widetilde{X}'$ be CW-complexes, and let $g : \widetilde{X} \to X$ and $f : \widetilde{X}' \to X'$ be homotopy equivalences with homotopy inverses $\tilde{g} : X \to \widetilde{X}$ and $h : X' \to \widetilde{X}'$, respectively, that is,

$$\tilde{g} \circ g \simeq \mathrm{id}_{\widetilde{X}}, \qquad g \circ \tilde{g} \simeq \mathrm{id}_X, \qquad h \circ f \simeq \mathrm{id}_{\widetilde{X}'}, \quad \text{and} \quad f \circ h \simeq \mathrm{id}_{X'}.$$

All mappings induce isomorphisms in cohomology.

Let A, A', B be CW-complexes such that $A \cap B = A' \cap B$ and is a CW-subcomplex of A, A', and B. Let $\widetilde{X} = A \cup B$ and $\widetilde{X}' = A' \cup B$ carry the induced CW-structures. By i^*, j^*, i'^*, j'^* we denote the homomorphisms of the long exact cohomology sequences of the pairs $(\widetilde{X}, B)$, $(\widetilde{X}, A)$, (X', B), or (X', A'), respectively. Now we obtain the following commutative diagram. Here the cohomology has coefficients in a group G which is not quoted in the diagram. Again exc and exc$'$ denote the obvious excision isomorphisms.

1.11. Diagram.

$$
\begin{tikzcd}
H^p(X') \arrow[d, "f^*"', "\cong"] & \times & H^q(X') \arrow[r] \arrow[d, "f^*"', "\cong"] & H^{p+q}(X') \arrow[d, "f^*"', "\cong"] & (\Phi', \Psi') \arrow[r, mapsto] \arrow[d, mapsto] & \Phi' \cup \Psi' \arrow[d, mapsto] \\
H^p(\widetilde{X}') & \times & H^q(\widetilde{X}') \arrow[r] & H^{p+q}(\widetilde{X}') & (\widetilde{\Phi}', \widetilde{\Psi}') \arrow[r, mapsto] & \tilde{\Phi}' \cup \tilde{\Psi}' \\
H^p(\widetilde{X}', A') \arrow[u, "j'^*"] \arrow[d, "\mathrm{exc}'"', "\cong"] & \times & H^q(\widetilde{X}') \arrow[r] \arrow[u, "="'] \arrow[d, "i'^*"'] & H^{p+q}(\widetilde{X}', A') \arrow[u, "j'^*"] \arrow[d, "\mathrm{exc}'"', "\cong"] & (\widetilde{\Phi}', \widetilde{\Psi}') \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \widetilde{\Phi}' \cup \widetilde{\Psi}' \arrow[u, mapsto] \arrow[d, mapsto] \\
H^p(B, B \cap A') \arrow[d, equal] & \times & H^q(B) \arrow[r] \arrow[d, equal] & H^{p+q}(B, B \cap A') \arrow[d, equal] & (\Phi|B, \Psi|B) \arrow[r, mapsto] \arrow[d, equal] & \Phi|B \cup \Psi|B \arrow[d, equal] \\
H^p(B, B \cap A) & \times & H^q(B) \arrow[r] & H^{p+q}(B, B \cap A) & (\Phi|B, \Psi|B) \arrow[r, mapsto] & \Phi|B \cup \Psi|B \\
H^p(\widetilde{X}, A) \arrow[u, "\mathrm{exc}"', "\cong"] \arrow[d, "j^*"'] & \times & H^q(\widetilde{X}) \arrow[r] \arrow[u, "i^*"] \arrow[d, "="'] & H^{p+q}(\widetilde{X}, A) \arrow[u, "\mathrm{exc}"', "\cong"] \arrow[d, "j^*"'] & (\widetilde{\Phi}_r, \widetilde{\Psi}) \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \widetilde{\Phi}_r \cup \widetilde{\Psi} \arrow[u, mapsto] \arrow[d, mapsto] \\
H^p(\widetilde{X}) & \times & H^q(\widetilde{X}) \arrow[r] & H^{p+q}(\widetilde{X}) & (\widetilde{\Phi}, \widetilde{\Psi}) \arrow[r, mapsto] & \widetilde{\Phi} \cup \widetilde{\Psi} \\
H^p(X') \arrow[u, "g^*"', "\cong"] & \times & H^q(X') \arrow[r] \arrow[u, "g^*"', "\cong"] & H^{p+q}(X') \arrow[u, "g^*"', "\cong"] & (\Phi', \Psi') \arrow[r, mapsto] \arrow[u, mapsto] & \Phi' \cup \Psi' \arrow[u, mapsto]
\end{tikzcd}
$$

Consider the following conditions.

(0) $\Phi \in H^p(X)$ and $\Psi \in H^q(X)$ such that Φ vanishes on A.
(1) $\mathrm{exc}'(\ker j'^*) = \mathrm{exc}(\ker j^*)$ and j'^* is surjective in dimension $p+q$.
(1′) j'^* is injective in dimension $p+q$.
(2) $i'^*(H^q(\widetilde{X}')) = i^*(H^q(\widetilde{X}))$.
(3) $f : \widetilde{X}' \to X'$ is a homotopy equivalence.
(4) $g : \widetilde{X} \to X$ is a homotopy equivalence.

As mentioned above, similar to the way Theorem 1.8 was proved from Diagram 1.7, we now have proved

1.12. Theorem. *If* (0), (2–4) *and* (1) *or* (1′) *are valid then, to given Φ and Ψ, there are cohomology classes $\Phi' \in H^p(X')$ and $\Psi' \in H^q(X')$ such that*

$$\Phi' \in \left[(f^*)^{-1} \circ j'^* \circ (\mathrm{exc}')^{-1} \circ \mathrm{exc} \circ (j^*)^{-1} \circ g^*\right](\Phi),$$

$$\Psi' \in \left[(f^*)^{-1} \circ \left(i'^*\right)^{-1} \circ i^* \circ g^*\right](\Psi) \quad \textit{and}$$

$$\Phi \cup \Psi = \left[(g^*)^{-1} \circ j^* \circ (\mathrm{exc})^{-1} \circ \mathrm{exc}' \circ \left(j'^*\right)^{-1} \circ f^*\right]\left(\Phi' \cup \Psi'\right).$$

2. ∪-Products of Seifert manifolds

In this section we describe “small” CW-complexes of Seifert fibre spaces, following [1], [3, §4.2], [5,9]. For the definition of Seifert manifolds see [8,14].

2.1. A fibered solid torus. Let us shortly repeat the definition of a Seifert fibration. Let $V = S^1 \times D^2$ be a solid torus where the boundary $S^1 \times S^1$ is fibered by the circles "parallel" to $\{(e^{2\pi i t/\alpha}, e^{2\pi i\beta t/\alpha}) \mid 0 \leqslant t \leqslant 1\}$ where α, β are relatively prime integers, $\alpha > 0$. This fibration is extended radially to the interior of V. This defines a locally trivial fibration of $V \setminus (S^1 \times \{0\})$. But there is also the central circle $S^1 \times \{0\}$. If $\alpha > 1$ this central circle and V are respectively called an *exceptional fibre* and *fibred solid torus of type* (α, β). Clearly, every fibre in the interior of the fibred solid torus V except the central one is the central fibre of a locally trivially fibred solid torus the fibres of which are from the considered fibration of V.

2.2. Definition. Let M^3 be a 3-manifold fibred into circles such that every fibre h admits a regular neighborhood on which the fibering induces the structure of a fibred solid torus with h as central fibre. Then M^3 is called a *Seifert manifold*. Let us call two points of M^3 equivalent if they belong to the same fibre. Then the quotient space Z, called the base, is homeomorphic to a surface and the projection defines a locally trivial fibration except at the exceptional fibres. We make the assumption that

(a) M^3 and the base $Z = Z_g$ are orientable,
(b) the genus of the base $Z = Z_g$ is g,
(c) an obstruction to the existence of a "section", the so-called *Euler number* $b \in \mathbb{Z}$,
(d) there are r exceptional fibres of the types $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$.

Then M^3 is denoted by $SF(g, b, r\colon (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$.

The classification with respect to fibre preserving homeomorphisms has been done by Seifert [10] and it turned out that in most cases this is also the topological classification of the manifolds (without taking care of the fibre structure), see [13,8]. Next we describe a "small" cell decomposition of a Seifert manifold.

2.3. A cell decomposition of the base. On the *base surface* Z_g we fix a point e^0 as single 0-cell, $(2g + r + 1)$ arcs $q_0^1, q_1^1, \dots, q_r^1, t_1^1, \dots, t_{2g}^1$ starting and ending at e^0 and otherwise pairwise disjoint as 1-cells, and $(r + 2)$ 2-cells $D_1^2, \dots, D_r^2, D_0^2, e^2$ where q_j^1 is the boundary of D_j^2 and

$$\partial e^2 = t_1^1 t_2^1 (t_1^1)^{-1} (t_2^1)^{-1} \cdots t_{2g-1}^1 t_{2g}^1 (t_{2g-1}^1)^{-1} (t_{2g}^1)^{-1} q_1^1 \cdots q_r^1 q_0^1.$$

(This is close to a standard complex of a surface with $r + 1$ exceptional points.) Define $Z_g^* = \overline{Z_g \setminus (D_1^2 \cup \cdots \cup D_r^2 \cup D_0^2)}$.

2.4. A cell decomposition of the locally trivial central part. To get to M^3, we "multiply the cells of the base by a fibre". We add a 1-cell h^1, a curve starting and ending at e^0, $(2g + r + 1)$ 2-cells

$$t_j^2 = t_j^1 \times h^1, \quad 1 \leqslant j \leqslant 2g, \qquad q_k^2 = q_k^1 \times h^1, \quad 1 \leqslant k \leqslant n, \qquad q_0^2 = q_0^1 \times h^1,$$

and one 3-cell $e^3 := e^2 \times h^1$. The 3-space M_0 so far obtained has $(r+1)$ tori as boundary components and admits a locally trivial fibration over $\overline{Z_g \setminus (D_0^2 \cup \cdots \cup D_r^2)}$ with the circle as fibre.

2.5. Solid tori containing the exceptional fibres. We attach solid tori $V_k = V_k(\alpha_k, \beta_k)$, $1 \leqslant k \leqslant n$, $V_0 = V_0(1, b)$ to the boundary components. In the boundary of V_i $(0 \leqslant i \leqslant r)$ we choose a fibre h_i^1, a crosscut q_i^1, and a meridian $m_i^1 \simeq (q_i^1)^{\alpha_i}(h_i^1)^{\beta_i}$ for $i > 0$ and $m_0 \simeq q_0^1(h_0^1)^b$, homotopy taken in ∂V_i. Now V_i is a CW-complex with a 0-cell $e_i^0 = q_i^1 \cap h_i^1$, two 1-cells q_i^1, h_i^1, two 2-cells q_i^2, m_i^2 with

$$\partial q_i^2 = q_i^1 h_i^1 \bigl(q_i^1\bigr)^{-1}\bigl(h_i^1\bigr)^{-1}, \qquad \partial m_i^2 \simeq \bigl(q_i^1\bigr)^{\alpha_i}\bigl(h_i^1\bigr)^{\beta_i}, \qquad \partial m_0^2 \simeq q_0^1\bigl(h_0^1\bigr)^b,$$

and a 3-cell m_i^3 the boundary of which consists of the "cylinder" q_i^2 and two copies of the disc m_i^2. Now the boundary of V_i is identified with the ith component of the boundary of M_0. Here e_i^0 is identified with e^0, q_i^1 from V_i with q_i^1 from the ith boundary torus of M_0, and similar h_i^1 and q_i^2 from V_i with h^1 and q_i^2 from M_0. So we get a CW-complex structure on M^3.

2.6. Proposition. *The Seifert manifold $M^3 = SF(g, b, r\colon (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ admits a* standard *CW-complex structure with*

$p = 0:$	e^0	*vertex in Z_g;*
$p = 1:$	$t_1^1, \ldots, t_{2g}^1$	*closed curves in Z_g,*
	$q_1^1, \ldots, q_r^1, q_0^1$	*boundary curves of Z_g^*,*
	h^1	*fibre;*
$p = 2:$	e^2	*section surface over Z_q^*,*
	$t_1^2, \ldots, t_{2g}^2$	*defining annuli in $S_g^* \times S^1$,*
	$q_1^2, \ldots, q_r^2, q_0^2$	*defining tori in ∂M_0,*
	$m_1^2, \ldots, m_r^2, m_0^2$	*meridian discs of the solid tori $V_1, \ldots, V_r, V_0$;*
$p = 3:$	e^3	*locally trivial fibred central part of M^3,*
	$m_1^3, \ldots, m_r^3, m_0^3$	*the interior of the solid tori $V_1, \ldots, V_r, V_0$.*

2.7. Standard chain complex of a Seifert manifold. The cells from 2.6 define the free generators for the cellular chain groups $C_p(M^3)$ of $M^3 = SF(g, b, r\colon (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ and we will denote them by the same letters as the cells:

dim 0 : e^0;

dim 1 : $t_1^1, \ldots, t_{2g}^1, q_1^1, \ldots, q_r^1, q_0^1, h^1$;

dim 2 : $e^2, t_1^2, \ldots, t_{2g}^2, q_1^2, \ldots, q_r^2, q_0^2, m_1^2, \ldots, m_r^2, m_0^2$;

dim 3 : $e^3, m_1^3, \ldots, m_r^3, m_0^3$.

The boundaries are obtained from 2.6:

$$\begin{array}{lll}
p=0: & \partial_0 e^0 = 0; & \\
p=1: & \partial_1 t_i^1 = 0 & \text{for } 1 \leqslant i \leqslant 2g, \\
 & \partial_1 q_k^1 = 0 & \text{for } 0 \leqslant k \leqslant r, \\
 & \partial_1 h^1 = 0; & \\
p=2: & \partial_2 e^2 = \sum_{k=0}^r q_k^1, & \\
 & \partial_2 t_i^2 = 0 & \text{for } 1 \leqslant i \leqslant 2g, \\
 & \partial_2 q_k^2 = 0 & \text{for } 0 \leqslant i \leqslant r, \\
 & \partial_2 m_k^2 = \alpha_k q_k^1 + \beta_k h^1 & \text{for } 1 \leqslant k \leqslant r, \\
 & \partial_2 m_0^2 = q_0^1 + b h^1; & \\
p=3: & \partial_3 e^3 = \sum_{k=0}^r q_k^2, & \\
 & \partial_3 m_k^3 = q_k^2 & \text{for } 0 \leqslant k \leqslant r.
\end{array}$$

Next we determine the corresponding cochains. For convenience we restrict ourselves to the case of cohomology with coefficients in $\mathbb{Z}_2$.

2.8. Standard cochain complex of a Seifert manifold. Define the cochain groups and the coboundaries by $C^p(M^3) = \mathrm{Hom}(C_p(M^3), \mathbb{Z}_2)$ and

$$\delta_p : C^p(M^3) \to C^{p+1}(M^3), \qquad \delta_p(\tilde{c}^p)(c^{p+1}) = \tilde{c}^p(\partial_{p+1} c^{p+1}).$$

Assume that the exceptional fibres are ordered such that

$$\alpha_k \equiv \begin{cases} 0 \bmod 2 & \text{for } 1 \leqslant k \leqslant n, \\ 1 \bmod 2 & \text{for } n+1 \leqslant k \leqslant r. \end{cases} \tag{1}$$

For the operation of $C^p(M^3)$ on $C_q(M^3)$ we use the scalar product form

$$\langle \tilde{c}^p, c_q \rangle = \begin{cases} \tilde{c}^p(c_q) & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Then to every generator x from 2.7 there corresponds a (uniquely determined) cochain $\tilde{x}$ such that for the generators y of $C_\bullet(M^3)$, see 2.7,

$$\langle \tilde{x}, y \rangle = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the coboundaries of the generators are:

$$\begin{array}{lll}
p=0: & \delta^0 \tilde{e}^0 = 0; & \\
p=1: & \delta^1 \tilde{t}_i^1 = 0 & \text{for } 1 \leqslant i \leqslant 2g, \\
 & \delta^1 \tilde{q}_k^1 = \tilde{e}^2 + \alpha_k \tilde{m}_k^2 = \tilde{e}^2 & \text{for } 1 \leqslant k \leqslant n, \\
 & \qquad = \tilde{e}^2 + \alpha_k \tilde{m}_k^2 = \tilde{e}^2 + \tilde{m}_k^2 & \text{for } n+1 \leqslant k \leqslant r, \\
 & \delta^1 \tilde{q}_0^1 = \tilde{e}^2 + \tilde{m}_0^2, &
\end{array}$$

$$
\begin{array}{lll}
 & \delta^1\tilde{h}^1 = b\tilde{m}_0^2 + \sum_1^r \beta_k \tilde{m}_k^2 & \\
 & \qquad = b\tilde{m}_0^2 + \sum_1^n \tilde{m}_k^2 + \sum_{n+1}^r \beta_k \tilde{m}_k^2; & \\
p=2: & \delta^2\tilde{e}^2 = 0, & \\
 & \delta^2\tilde{t}_i^2 = 0 & \text{for } 1 \leqslant i \leqslant 2g, \\
 & \delta^2\tilde{q}_k^2 = \tilde{e}^3 + \tilde{m}_k^3 & \text{for } 0 \leqslant k \leqslant r, \\
 & \delta^2\tilde{m}_k^2 = 0 & \text{for } 0 \leqslant k \leqslant r; \\
p=3: & \delta^3\tilde{e}^3 = 0, & \\
 & \delta^3\tilde{m}_k^3 = 0 & \text{for } 0 \leqslant k \leqslant r.
\end{array}
$$

We will check this for the case $\delta^1\tilde{h}^1$. From 2.7 we obtain that the m_k^2, $1 \leqslant k \leqslant r$ are the only generating chains which are bounded by h^1 and

$$
\langle \delta^1\tilde{h}^1, m_k^2\rangle = \langle \tilde{h}^1, \partial_2 m_k^2\rangle = \begin{cases} b & \text{if } k=0, \\ \beta_k & \text{for } 1 \leqslant k \leqslant r, \end{cases}
$$

and this implies

$$
\delta^1\tilde{h}^1 = b\tilde{m}_0^2 + \sum_1^r \beta_k \tilde{m}_k^2.
$$

Next we give a description of the cohomology ring of M^3.

2.9. Theorem. *Let $M^3 = SF(g,b,r\colon\ (\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ be a Seifert manifold fulfilling the conditions* (1) *and define $\beta = b + \beta_1 + \cdots + \beta_r$.*

(a) *The cohomology groups of M^3 with coefficients in $\mathbb{Z}_2$ are*

$$
\begin{aligned}
H^0(M^3;\mathbb{Z}_2) &\cong H^3(M^3;\mathbb{Z}_2) \cong \mathbb{Z}_2, \\
H^1(M^3;\mathbb{Z}_2) &\cong H^2(M^3;\mathbb{Z}_2) \\
&\cong \begin{cases} \mathbb{Z}_2^{2g+n-1} & \text{if } n>0, \\ \mathbb{Z}_2^{2g+1} & \text{if } n=0 \text{ and } \beta \equiv 0 \bmod 2, \\ \mathbb{Z}_2^{2g} & \text{if } n=0 \text{ and } \beta \equiv 1 \bmod 2. \end{cases}
\end{aligned}
$$

(b) *Generators for the cohomology groups are*

$$
\begin{array}{lll}
\dim 0: & 1 = \{\tilde{e}^0\}; & \\
\dim 1: & \Psi_i' = \{\tilde{t}_{2i-1}^1\} & 1 \leqslant i \leqslant g, \\
 & \Psi_i'' = \{\tilde{t}_{2i}^1\} & 1 \leqslant i \leqslant g, \\
 & \Phi_k = \{\tilde{q}_k^1 + \tilde{q}_n^1\} & 1 \leqslant k \leqslant n-1 \text{ if } n>0, \\
 & \Phi = \{\tilde{\varphi}\} & \text{with } \tilde{\varphi} = \tilde{h}^1 + b\tilde{q}_0^1 + \sum_1^r \beta_k \tilde{q}_k^1 \\
 & & \text{if } n=0 \text{ and } \beta \equiv 0 \bmod 2;
\end{array}
$$

$$\begin{array}{lll}
\dim 2: & \Gamma_i' = \{\tilde{t}_{2i-1}^2\} & 1 \leqslant i \leqslant g, \\
& \Gamma_i'' = \{\tilde{t}_{2i}^2\} & 1 \leqslant i \leqslant g, \\
& \Lambda_k = \{\tilde{m}_k^2\} & 1 \leqslant k \leqslant n-1 \text{ if } n > 0, \\
& \Lambda = \{\tilde{e}^2\} = \{\tilde{m}_k^2\} & 0 \leqslant k \leqslant r, \text{ if } n = 0,\ \beta \equiv 0 \bmod 2; \\
\dim 3: & \Theta = \{\tilde{e}^3\} = \{\tilde{m}_k^3\} & 0 \leqslant k \leqslant r.
\end{array}$$

(c) *The* $\cup$*-products in* $H^*(M^3; \mathbb{Z}_2)$ *are given for the generators of* (b). *Let* δ_{ij} *denote the Kronecker symbol. The following table contains all non-trivial* $\cup$*-products between the generators of* $H^*(M^3; \mathbb{Z}_2)$ *from* (b), *with the exception of the products with the unity element* 1. *For this we have* $1 \cup \Omega = \Omega$ *for every element* Ω. *In the following the subscripts* i, j *vary from* 1 *to* g, k, l *from* 1 *to* r.

$$\begin{array}{ll}
\textit{Case } n > 0: & \Phi_k \cup \Phi_l = \delta_{kl}\binom{\alpha_k}{2}\Lambda_k + \binom{\alpha_n}{2}\Lambda_n \quad \textit{with} \\
& \Lambda_n = \sum_{k=1}^{n-1} \Lambda_k = \sum_{k=1}^{n-1}\{\tilde{m}_k^2\} = \{\tilde{m}_n^2\}, \\
& \Psi_i' \cup \Gamma_j'' = \delta_{ij}\Theta, \\
& \Psi_i'' \cup \Gamma_j' = \delta_{ij}\Theta, \\
& \Phi_k \cup \Lambda_l = \delta_{kl}\Theta, \\
\textit{Case } n = 0, & \\
\beta \equiv 1 \bmod 2: & \Psi_i' \cup \Gamma_j'' = \delta_{ij}\Theta, \\
& \Psi_i'' \cup \Gamma_j' = \delta_{ij}\Theta, \\
\textit{Case } n = 0, & \\
\beta \equiv 0 \bmod 2: & \Psi_i'' \cup \Psi_j' = \delta_{ij}\Lambda, \\
& \Psi_i' \cup \Phi = \Gamma_i', \\
& \Psi_i'' \cup \Phi = \Gamma_i'', \\
& \Phi \cup \Phi = \begin{cases} \Lambda & \textit{if } b - 2p + \sum_{k=m+1}^{r} \alpha_k\beta_k \equiv 2 \bmod 4, \\ 0 & \textit{otherwise}, \end{cases} \\
& \Psi_i' \cup \Gamma_j'' = \delta_{ij}\Theta, \\
& \Psi_i'' \cup \Gamma_j' = \delta_{ij}\Theta, \\
& \Phi \cup \Lambda = \Theta.
\end{array}$$

In the last case $(n = 0)$ *we assume that*

$$\begin{array}{ll}
\beta_k \equiv 2 \bmod 4 & \textit{for } 1 \leqslant k \leqslant p, \\
\beta_k \equiv 0 \bmod 4 & \textit{for } p+1 \leqslant k \leqslant m, \\
\beta_k \equiv 1 \bmod 2 & \textit{for } m+1 \leqslant k \leqslant r.
\end{array}$$

Remarks to the Proof. For $g = 0$ the cohomology ring $H^*(M^3; \mathbb{Z}_2)$ has been determined in [2,3] using free resolutions of the $\mathbb{Z}\pi$-module $\mathbb{Z}$, $\pi = \pi_1(M^3)$. But already in this case the formulae become quite long, and this unpleasant behavior enormously increases for the general case, see [4]. To avoid it, the notion of a cohomology class "vanishing on subsets" has been introduced in [1] and the cohomology ring for all Seifert manifolds has been determined. The approach in [1] was based on cells and chains. Starting in discussions with

André Legrand, the method of [1] has been translated into Diagram 1.7, see [7], Section 6. Theorem 2.9 in this form can be found in [5].

For the proof, many different suitable decompositions of Seifert manifolds have to be considered. To get an impression of the method see [1], Example 1.9 and 5.6.

2.10. On Seifert manifolds with boundaries. The graph manifolds studied in the Sections 3–5 are obtained as unions of bounded Seifert manifolds which have a similar cell decomposition to the closed ones. The only difference is that not all boundary tori of M_0, see 2.4, are used for attaching solid tori corresponding to exceptional fibres or to the section obstruction. Instead of $(1+r)$ tori we have to take $(1+r+\varrho)$ boundary components of M_0 to add solid tori as in 2.5 for $k=0,1,\ldots,r$ and ϱ components to build ∂M^3. On each of these boundary tori the fibration of M_0 defines a S^1-fibration over S^1 and e^2 defines a crosscut. So we obtain two simple closed curves h^1 and q^1 which intersect in a vertex and generate a basis of the fundamental group of the torus. We will use this situation in Definition 3.2. See also 3.7.

3. Three-dimensional graph manifolds

Graph manifolds are defined and classified by Waldhausen [13]. They appear also in the approach of A.T. Fomenko and others to integrable Hamiltonian systems, see [6]; for details see also [5,9]. In this section we describe CW-complexes for a graph manifold W^3 using the cell decomposition from Section 2 and we determine a cochain complex for W^3.

3.1. Definition. Let W^3 be a compact orientable 3-manifold with or without boundary. Let $\mathcal{T}$ be a finite system of disjoint tori $T^2_{\tau,t;\gamma}$ lying in $\mathring{W}^3$, and let $U(\mathcal{T})$ be a (closed) regular neighborhood of $\mathcal{T}$ in W^3. If each component of $\overline{W^3\setminus U(\mathcal{T})}$ is a bounded Seifert manifold SF^∂_τ, the manifold W^3 is called a *graph manifold*, $\mathcal{T}$ is *a decomposing system of tori*, and $\mathcal{T}$ *defines a graph structure on* W^3. The subscript triple $(\tau,t;\gamma)$ represents a torus lying between the Seifert components SF^∂_τ and SF^∂_t. The index γ runs from 1 to the number of tori directed from SF^∂_τ to SF^∂_t. The tori and their regular neighborhoods are called the *gluing tori* and *gluing cylinders*, respectively, of W^3 or—more precisely—of $\mathcal{T}$.

In the following we consider *closed* graph manifolds. The components of $\overline{W^3\setminus U(\mathcal{T})}$ will be called the *Seifert components* of (the graph structure on) W^3. We often close the Seifert manifolds SF^∂_τ with boundary by solid tori and extend the fibration by a trivial one on the solid torus to obtain closed Seifert manifolds SF_τ. Let N be the number of Seifert components: $SF_1,\ldots,SF_N$. To come back from the SF_τ's to the SF^∂_τ's and W^3, we first drill off from the SF_τ's solid tori on which the Seifert fibrations of the SF_τ induce trivial fibrations. Next we glue together different boundary tori of the SF^∂_τ's using cylinders of the form $T^2\times[0,1]$. The obtained manifold is fibre-preserving homeomorphic to the original graph manifold W^3.

Notice that SF_τ is not a subspace of W^3.

3.2. Definition. Every decomposing torus $T_{\tau,t;\gamma}\in\mathcal{T}$ defines a homeomorphism between the two components of $\partial U(T_{\tau,t;\gamma})$ from $\partial SF^\partial_\tau$ and ∂SF^∂_t. On both boundary components we get—according to 2.10—the fibre and a crosscut in the obvious way. We express the images of these curves ν_τ, h_τ from the torus from SF_τ^∂ by the curves ν_t, h_t from SF_t^∂ and obtain

$$\nu_\tau \sim a_{\tau,t;\gamma}\nu_t + b_{\tau,t;\gamma}h_t, \quad h_\tau \sim c_{\tau,t;\gamma}\nu_t + d_{\tau,t;\gamma}h_t \quad \text{with } \begin{vmatrix} a_{\tau,t;\gamma} & b_{\tau,t;\gamma} \\ c_{\tau,t;\gamma} & d_{\tau,t;\gamma} \end{vmatrix} = 1,$$

homology taken on the considered torus and identifying curves with the same notation. To the gluing, thus, corresponds a matrix

$$A_{\tau,t;\gamma} = \begin{pmatrix} a_{\tau,t;\gamma} & b_{\tau,t;\gamma} \\ c_{\tau,t;\gamma} & d_{\tau,t;\gamma} \end{pmatrix}.$$

For the calculation of $H^*(W^3;\mathbb{Z}_2)$ we can reduce the coefficients mod 2 and take only values 0, 1 such that $\det A_{\tau,t;\gamma} \equiv 1 \bmod 2$; there are left only the following six matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We will not explicitly do the reduction, but will write (α,β) as before or $(\alpha,\beta) \bmod 2$ if this makes it clearer.

3.3. Theorem and Definition. *An orientable graph manifold is determined by a family of Seifert manifolds $(SF_\tau)_{1\leqslant\tau\leqslant N}$ and a finite list of homeomorphisms $\mathcal{A} = (A_{\tau,t;\gamma})_{(\tau,t;\gamma)\in I}$ between the decomposing tori*; *here*

$$I = \big\{(\tau,t;\gamma) \mid 1 \leqslant \tau,\ t \leqslant N,\ 1 \leqslant \gamma \leqslant \Gamma_{\tau,t}\big\}$$

where $\Gamma_{\tau,t} \geqslant 0$ counts the number of tori between SF_τ and SF_t, directed from the first to the second.

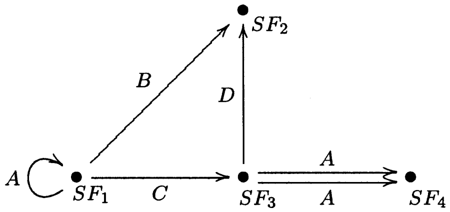
Next we use graphs to represent graph manifolds. A (directed) graph consists of vertices and directed edges. Every edge has a source and a target vertex which may coincide. To every edge σ there exists the inversely oriented edge leading from the target of σ to the source. In the following we write down only one of inversely directed edges.

3.4. Definition and Theorem. *A graph manifold W^3 admits a* presentation as a graph *$\mathcal{G}(W^3)$ where each Seifert component SF_τ of W^3 is represented by a vertex and every gluing cylinder by an edge of $\mathcal{G}(W^3)$. The vertices are marked by the component names SF_τ or by its invariants g_τ, b_τ, r_τ and the ingredients of the exceptional fibres. The γth edge oriented from SF_τ to SF_t is denoted by the matrix $A_{\tau,t;\gamma}$. In case of the unit matrix we do not indicate the matrix and, vice versa, if an edge is not marked by a matrix it represents a trivial gluing corresponding to the unit matrix. The graph $\mathcal{G}(W^3)$ determines the graph structure and, hence, the topological type of W^3.*

3.5. Example. The graph manifold

$$W = \Big(SF_1(g_1,b_1,r_1),\, SF_2(g_2,b_2,r_2),\, SF_3(g_3,b_3,r_3),\, SF_4(g_4,b_4,r_4);$$
$$\Big\{ A = A_{1,1;1} = A_{3,4;1} = A_{3,4;2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = A_{1,2;1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$C = A_{1,3;1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ D = A_{3,2;1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Big\} \Big)$$

is represented by the graph



3.6. Cell decomposition of a gluing cylinder. For every gluing

$$A_{\tau,t;\gamma} = \begin{pmatrix} a_{\tau,t;\gamma} & b_{\tau,t;\gamma} \\ c_{\tau,t;\gamma} & d_{\tau,t;\gamma} \end{pmatrix}$$

of the Seifert component SF_τ with the Seifert component SF_t there is the gluing cylinder $U(T_{\tau,t;\gamma})$ consisting of the following cells:

dim 0 :	e^0_τ, e^0_t;	
dim 1 :	h^1_τ, h^1_t	fibres of SF_τ and SF_t, respectively,
	$\mu^1_{\tau,t;\gamma}, \nu^1_{\tau,t;\gamma}$	cross cuts on the tori from ∂SF_τ and ∂SF_t, respectively,
	$\varepsilon^1_{\tau,t;\gamma}$	interval joining e^0_τ and e^0_t;
dim 2 :	$\mu^2_{\tau,t;\gamma}, \nu^2_{\tau,t;\gamma}$	2-cells on tori in ∂SF_τ and ∂SF_t, respectively,
	$\varepsilon^2_{\tau,t;\gamma}, \eta^2_{\tau,t;\gamma}$	discs in $U(T_{\tau,t;\gamma})$ corresponding to annuli between pairs of curves on the comp. of $\partial U(T_{\tau,t;\gamma})$;
dim 3 :	$\varepsilon^3_{\tau,t;\gamma}$.	

All 1-cells as well as the "torus" 2-cells $\mu^2_{\tau,t;\gamma}$, $\nu^2_{\tau,t;\gamma}$ are "closed", that is, correspond to closed curves or tori. The other boundaries are

$$\partial_2\varepsilon^2_{\tau,t;\gamma} = \mu^1_{\tau,t;\gamma} - a_{\tau,t;\gamma}\nu^1_{\tau,t;\gamma} - b_{\tau,t;\gamma}h^1_t,$$
$$\partial_2\eta^2_{\tau,t;\gamma} = h^1_\tau - c_{\tau,t;\gamma}\nu^1_{\tau,t;\gamma} - d_{\tau,t;\gamma}h^1_t,$$
$$\partial_3\varepsilon^3_{\tau,t;\gamma} = \mu^2_{\tau,t;\gamma} - \nu^2_{\tau,t;\gamma}.$$

In $\partial_3\varepsilon^3_{\tau,t;\gamma}$ each one of the discs $\varepsilon^2_{\tau,t;\gamma}$ and $\eta^2_{\tau,t;\gamma}$ is counted twice with different signs. Putting together the standard decompositions of the Seifert manifolds 2.7 and of the gluing cylinders 3.6 we obtain:

3.7. Theorem. *A graph manifold W^3 consists of a finite family of Seifert manifolds*

$$(SF_\tau)_{1\leqslant\tau\leqslant N} = \big(SF_\tau\big(g_\tau, b_\tau, r_\tau\colon\ (\alpha_{\tau 1},\beta_{\tau 1}),\dots,(\alpha_{\tau r_\tau},\beta_{\tau r_\tau})\big)\big)_{1\leqslant\tau\leqslant N}$$

and a finite list of gluings of tori in the Seifert manifolds by homeomorphisms

$$\mathcal{A}=\left\{A_{\tau,t;\gamma}=\begin{pmatrix} a_{\tau,t;\gamma} & b_{\tau,t;\gamma} \\ c_{\tau,t;\gamma} & d_{\tau,t;\gamma}\end{pmatrix}\;\middle|\; 1\leqslant\tau,\ t\leqslant N,\ 1\leqslant\gamma\leqslant\Gamma_{\tau,t}\right\}.$$

Hence W^3 is determined by $((SF_\tau)_{1\leqslant\tau\leqslant N};\mathcal{A})$. The number of solid tori cut off from the Seifert component SF_τ can be determined from the list $\mathcal{A}$ of gluings. It is necessary to drill $2|\mathcal{A}|$ tubes into the Seifert components SF_τ.

To get a cell decomposition of W^3 we take the CW-complexes of the SF_τ as given in 2.6 and add boundary tori corresponding to the gluing cylinders. If the torus represents the "departing", i.e., the τ-component of a gluing cylinder, then the cell $\mu^1_{\tau,t;\gamma}$, corresponding to a q^1_j in 2.7 or 2.10, is also in the boundary of the 2-cell e^2_τ of the base. Moreover the cell $\mu^2_{\tau,t;\gamma}=\mu^1_{\tau,t;\gamma}\times h^1_\tau$, describing the boundary torus, is part of the boundary of e^3_τ. The boundaries of both cells "vanish algebraically". If the torus realizes the "arrival", i.e., the t-component of a gluing cylinder, then e^2_τ is also bounded by the cell $\nu^1_{t',\tau;\gamma}$ for suitable t', γ and, thus, e^3_τ is also bounded by the cell $\nu^2_{t',\tau;\gamma}=\nu^1_{t',\tau;\gamma}\times h^1_\tau$.

Now we take these complexes of the Seifert manifolds and add the complexes of the gluing cylinders. Since we used the same notation for some cells like $\mu^1_{\tau,t;\gamma}$, $\mu^2_{\tau,t;\gamma}$, $\nu^1_{t',\tau;\gamma}$, $\nu^2_{t',\tau;\gamma}$ the identifications are already indicated.

3.8. Standard cell decompositions of graph manifolds. According to 3.7, 3.6 and 2.6, see also 2.7, we obtain the following cell decomposition for the graph manifold $W^3=((SF_\tau)_{1\leqslant\tau\leqslant N};\mathcal{A})$. Moreover we get free generators for their chain groups $C_p(W^3)$. In the following we denote the *base surface of SF_τ by Z_τ*, the variables τ and t run from 1 to N and γ covers all possibilities according to the two subscripts τ, t.

$p=0:$	e^0_τ	vertex in Z_τ;
$p=1:$	$t^1_{\tau1},\ldots,t^1_{\tau,2g_\tau}$	closed curves in Z_τ,
	$q^1_{\tau0},\ldots,q^1_{\tau,r_\tau}$	boundaries of Z_τ,
	h^1_τ	normal fibre in SF_τ,
	$\mu^1_{\tau,t;\gamma}$	in ∂Z_τ for "departing" gluing cylinders,
	$\nu^1_{t',\tau;\gamma}$	in ∂Z_τ for "arriving" cylinders,
	$\varepsilon^1_{\tau,t;\gamma}$	interval in gluing cylinder;
$p=2:$	e^2_τ	2-cell of Z_τ with "holes",
	$t^2_{\tau1},\ldots,t^2_{\tau,2g_\tau}$	from $t^1_{\tau j}\times h^1_\tau$,
	$q^2_{\tau0},\ldots,q^2_{\tau,r_\tau}$	boundary components of $Z_\tau\times S^1$,
	$m^2_{\tau0},\ldots,m^2_{\tau,r_\tau}$	meridian discs for the solid tori $V_{\tau j}$,
	$\varepsilon^2_{\tau,t;\gamma}$	2-cells in the cylinders $\partial U(T_{\tau,t;\gamma})$ inducing
	$\eta^2_{\tau,t;\gamma}$	the gluing homeomorphisms $A_{\tau,t;\gamma}$,
	$\mu^2_{\tau,t;\gamma}$	boundary torus of $\partial U(T_{\tau,t;\gamma})$ and of SF_τ,
	$\nu^2_{t',\tau;\gamma}$	boundary torus of $\partial U(T_{t',\tau;\gamma})$ and SF_τ;

$$\begin{array}{lll} p=3: & e^3_\tau & \text{“interior” of } SF_\tau\text{: } Z_\tau \times h^1_\tau, \\ & m^3_{\tau 0}, \ldots, m^3_{\tau, r_\tau} & \text{“interior” of the solid tori } V_{\tau j}, \\ & \varepsilon^3_{\tau,t;\gamma} & \text{“interior” of the gluing cylinder } U(T_{\tau,t;\gamma}). \end{array}$$

The boundary operators $\partial_p : C_p(W^3) \to C_{p-1}(W^3)$ are obtained from 2.7 and 3.2. From the generators of the chain complex we obtain generators for the cochains $\mathrm{Hom}(C_p, \mathbb{Z}_2)$ which we again denote by using a tilde. The coboundary operators are defined by

$$\delta_p(\varphi) = \varphi \circ \partial_{p+1}, \quad \varphi \in \mathrm{Hom}\big(C_p(W), \mathbb{Z}_2\big),$$

and this allows one to calculate the coboundaries of the generators dual to the above cells.

3.9. Boundaries and coboundaries for the standard CW-complex of a graph manifold. For every $\tau \in \{1, \ldots, N\}$, we order the boundary components of SF_τ such that for some n_τ, $0 \leqslant n_\tau \leqslant r_\tau$,

$$\alpha_{\tau j} \equiv \begin{cases} 0 \bmod 2 & \text{for } 1 \leqslant j \leqslant n_\tau, \\ 1 \bmod 2 & \text{for } n_\tau + 1 \leqslant j \leqslant r_\tau. \end{cases}$$

Since the α and β for every pair are relatively prime, $\beta_{\tau j} \equiv 1 \bmod 2$ for $1 \leqslant j \leqslant n_\tau$, and we obtain for $\tau = 1, \ldots, N$, $i = 1, \ldots, 2g_\tau$ and $k = 1, \ldots, r_\tau$ the following boundaries and coboundaries:

$$\begin{array}{lll}
p=0: & \partial_0 e^0_\tau = 0, & \delta_0 \tilde{e}^0_\tau = 0, \\
p=1: & \partial_1 t^1_{\tau i} = 0, & \delta_1 \tilde{t}^1_{\tau i} = 0, \\
& \partial_1 q^1_{\tau 0} = 0, & \delta_1 \tilde{q}^1_{\tau 0} = \tilde{e}^2_\tau + \tilde{m}^2_{\tau 0}, \\
& \partial_1 q^1_{\tau k} = 0, & \delta_1 \tilde{q}^1_{\tau k} = \tilde{e}^2_\tau + \alpha_{\tau k} \tilde{m}^2_{\tau k}, \\
& \partial_1 h^1_\tau = 0, & \delta_1 \tilde{h}^1_\tau = b\tilde{m}^2_{\tau 0} + \sum_1^{r_\tau} \beta_{\tau k} \tilde{m}^2_{\tau k}, \\
& \partial_1 \mu^1_{\tau,t;\gamma} = 0, & \delta_1 \tilde{\mu}^1_{\tau,t;\gamma} = \tilde{e}^2_\tau + \tilde{\varepsilon}^2_{\tau,t;\gamma}, \\
& \partial_1 \nu^1_{t',\tau;\gamma} = 0, & \delta_1 \tilde{\nu}^1_{t',\tau;\gamma} = \tilde{e}^2_\tau + a_{\tau,t;\gamma} \tilde{\nu}^2_{\tau,t;\gamma}, \\
& \partial_1 \varepsilon^1_{\tau,t} = 0, & \delta_1 \tilde{\varepsilon}^1_{\tau,t} = 0, \\
p=2: & \partial_2 e^2_\tau = \sum_0^{r_\tau} q^1_k + \sum \mu^1_{\tau,t;\gamma} & \delta_2 \tilde{e}^2_\tau = 0, \\
& \qquad + \sum \nu^1_{t',\tau;\gamma}, \quad (*) & \\
& \partial_2 t^2_{\tau i} = 0, & \delta_2 \tilde{t}^2_{\tau i} = 0, \\
& \partial_2 q^2_{\tau 0} = 0, & \delta_2 \tilde{q}^2_{\tau 0} = \tilde{e}^3_\tau + \tilde{m}^3_{\tau 0}, \\
& \partial_2 q^2_{\tau k} = 0, & \delta_2 \tilde{q}^2_{\tau k} = \tilde{e}^3_\tau + \tilde{m}^3_{\tau k}, \\
& \partial_2 m^2_{\tau 0} = q^1_{\tau 0} + b_\tau h^1_\tau, & \delta_2 \tilde{m}^2_{\tau 0} = 0, \\
& \partial_2 m^2_{\tau k} = \alpha_{\tau k} q^1_{\tau k} + \beta_{\tau k} h^1_\tau, & \delta_2 \tilde{m}^2_{\tau k} = 0, \\
& \partial_2 \mu^2_{\tau,t;\gamma} = 0, & \delta_2 \tilde{\mu}^2_{\tau,t;\gamma} = \tilde{e}^3_\tau + \tilde{\varepsilon}^3_{\tau,t;\gamma}, \\
& \partial_2 \nu^2_{t',\tau;\gamma} = 0, & \delta_2 \tilde{\nu}^2_{t',\tau;\gamma} = \tilde{e}^3_\tau + \tilde{\varepsilon}^3_{t',\tau;\gamma}, \\
& \partial_2 \varepsilon^2_{\tau,t;\gamma} = \mu_{\tau,t;\gamma} - a_{\tau,t;\gamma} \nu^1_{\tau,t;\gamma} & \delta_2 \tilde{\varepsilon}^2_{\tau,t;\gamma} = 0, \\
& \qquad - b_{\tau,t;\gamma} h^1_t, &
\end{array}$$

$$
\begin{array}{lll}
 & \partial_2\eta^2_{\tau,t;\gamma} = h^1_\tau - c_{\tau,t;\gamma}\nu^1_{\tau,t;\gamma} & \delta_2\tilde{\eta}^2_{\tau,t;\gamma} = 0, \\
 & \qquad - d_{\tau,t;\gamma}h^1_t, & \\
p = 3: & \partial_3 e^3_\tau = \sum_0^{r_\tau} q^2_{\tau k} + \sum \mu^2_{\tau,t;\gamma} & \delta_3\tilde{e}^3_\tau = 0, \\
 & \qquad + \sum \nu^2_{t',\tau;\gamma}, \quad (*) & \\
 & \partial_3 m^3_{\tau 0} = q^2_{\tau 0}, & \delta_3\tilde{m}^3_{\tau 0} = 0, \\
 & \partial_3 m^3_{\tau k} = q^2_{\tau k}, & \delta_3\tilde{m}^3_{\tau k} = 0, \\
 & \partial_3\varepsilon^3_{\tau,t;\gamma} = \mu^2_{\tau,t;\gamma} - \nu^2_{\tau,t;\gamma}, & \delta_3\tilde{\varepsilon}^3_{\tau,t;\gamma} = 0.
\end{array}
$$

In the equations marked by $(*)$, the sums are taken over all possible (t,γ) or (t',γ) for the τ in question.

4. The canonical graph of a graph manifold

Given a graph manifold W^3, the complexes described in Section 3 are, of course, not uniquely determined. For classification aims one tries to make the Seifert components as big as possible as done in [13,5]. In [6] the graph manifolds are decomposed in "small" elementary 3-manifolds of five or only two types, so-called *bricks*. These bricks are closely related to Hamiltonian systems, but they are not appropriate for our purposes.

We shall reduce the structure of graph manifolds to simple *canonical graphs* corresponding to a *canonical cell decomposition* which consists of so-called *building stones* defining the invariants from Section 3: genus, Euler class, number and type of exceptional fibers, gluing cylinders. Since we only calculate cohomology with coefficients in $\mathbb{Z}_2$ we could restrict ourselves to manifolds admitting only gluing matrices as given in 3.2 and the reader can think of all α, β as 0's or 1's.

4.1. Notation. For a graph manifold W^3, let sf_τ denote a Seifert component of type $SF_\tau(0,0,0)$, that is, it adds to W^3 a subspace of the form $\overline{S^2 \setminus D^2_1 \cup \cdots \cup D^2_r} \times S^1$ where the number r of boundary components has to be determined by the gluing invariants of the graph manifold.

4.2. Proposition and Definition. *The Seifert space* $SF(0,0,1\colon(\alpha,\beta))$ *is homeomorphic to the graph manifold*

$$
W_r = \left((sf_1, sf_2, sf_3); \left\{A_{1,2;1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_{3,2;1} = \begin{pmatrix} \alpha & \beta \\ * & * \end{pmatrix}\right\}\right),
$$

$$
\bullet_{SF(0,0,1\colon(\alpha,\beta))} \approx \bullet_{sf_1} \longrightarrow \bullet_{sf_2} \xleftarrow{\binom{\alpha\ \beta}{*\ *}} \bullet_{sf_3}.
$$

The row $(*\,*)$ *indicates a pair of integers such that the matrix has determinant* 1. *Here* sf_1 *and* sf_3 *have one while* sf_2 *has two boundary components. The CW-complex* W_r *is called the* building stone of an exceptional fiber.

Proof. The Seifert manifold is obtained from an annulus times a circle by attaching a trivial fibered solid torus and one of type (α, β). Every piece can be considered as a Seifert manifold with boundary. $\square$

By similar arguments one obtains the following results from [9].

4.3. Proposition. *The graph manifold*

$$\left(SF_1(g_1, b_1, r_1); \left\{A_{1,1;1} = \begin{pmatrix} a_{1,1;1} & b_{1,1;1} \\ c_{1,1;1} & d_{1,1;1} \end{pmatrix}\right\}\right)$$

is homeomorphic to the graph manifold

$$\begin{aligned} W' = \Bigg(&\big(SF_1(g_1, b_1, r_1), sf_2, sf_3, sf_4\big); \\ &\left\{A_{1,2;1} = A_{2,3;1} = A_{2,4;1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_{3,4;1} = \begin{pmatrix} a_{1,1;1} & b_{1,1;1} \\ c_{1,1;1} & d_{1,1;1} \end{pmatrix}\right\}\Bigg). \end{aligned}$$

For $g_1 = b_1 = r_1 = 0$ we obtain the next building stone.

4.4. Corollary. *The graph manifold* $\big(sf_1; \big\{A_{1,1;1} = \left(\begin{smallmatrix} a_{1,1;1} & b_{1,1;1} \\ c_{1,1;1} & d_{1,1;1} \end{smallmatrix}\right)\big\}\big)$ *is homeomorphic to the graph manifold*

$$\begin{aligned} W_g = \Bigg(&(sf_1, sf_2, sf_3, sf_4); \\ &\left\{A_{1,2;1} = A_{2,3;1} = A_{2,4;1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_{3,4;1} = \begin{pmatrix} a_{1,1;1} & b_{1,1;1} \\ c_{1,1;1} & d_{1,1;1} \end{pmatrix}\right\}\Bigg). \end{aligned}$$

For its diagram see W_g in 4.9. *The CW-complex W_g is called a* building stone of a self gluing.

4.5. Proposition and Definition. *The graph manifold*

$$W^3 = \left((sf_1, sf_2); \left\{A_{1,2;1} = \begin{pmatrix} a_{1,2;1} & b_{1,2;1} \\ c_{1,2;1} & d_{1,2;1} \end{pmatrix}\right\}\right)$$

is homeomorphic to the graph manifold

$$\begin{aligned} W_v = \Bigg(&(sf_1, sf_2, sf_3, sf_4, sf_5, sf_6, sf_7, sf_8); \\ &\left\{A_{1,2;1} = A_{2,3;1} = A_{2,4;1} = A_{5,7;1} = A_{6,7;1} = A_{7,8;1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\right. \\ &\left.\ A_{4,5;1} = \begin{pmatrix} a_{1,2;1} & b_{1,2;1} \\ c_{1,2;1} & d_{1,2;1} \end{pmatrix}\right\}\Bigg). \end{aligned}$$

The manifold W_v is called a building stone of a gluing; *for its diagram see* 4.9.

4.6. Proposition. *The Seifert manifold* $SF(g,b,r\colon (\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ *is homeomorphic to the graph manifold*

$$W = \Bigl(\bigl(SF_1(1,0,0),\dots,SF_g(1,0,0),SF_{g+1}\bigl(0,0,1\colon (\alpha_1,\beta_1)\bigr),\dots,$$
$$SF_{g+r}\bigl(0,0,1\colon (\alpha_r,\beta_r)\bigr),SF_{g+r+1}(0,b,0)\bigr);$$
$$\left\{A_{\tau,\tau+1;1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \,\middle|\, 1 \leqslant \tau \leqslant g+r\right\}\Bigr).$$

The corresponding graph is

$$\bullet_{SF_1(1,0,0)} \longrightarrow \cdots \longrightarrow \bullet_{SF_g(1,0,0)} \longrightarrow \bullet_{SF_{g+1}(0,0,1\colon (\alpha_1,\beta_1))} \longrightarrow \cdots$$
$$\cdots \longrightarrow \bullet_{SF_{g+r}(0,0,1\colon (\alpha_r,\beta_r))} \longrightarrow \bullet_{SF_{g+r+1}(0,b,0)}.$$

4.7. Proposition. *The graph manifold*

$$W = \left((sf_1, sf_2);\left\{A_{1,2;1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ A_{1,2;2} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\}\right)$$

and the Seifert manifold $SF(1,0,1\colon (1,1))$ *are homeomorphic.*

4.8. Definition. A graph $\mathcal{G}(W)$ representing a standard cell decomposition of a closed orientable graph manifold is called a *canonical graph* if it has the following properties.

(P1) Every vertex has a marking $SF_\tau(0,0,0)$, that is, it is a Seifert component homeomorphic to $S^2 \times S^1$.
(P2) There are no parallel edges, that is, every $\gamma = 1$. Therefore we drop the subscript γ in the following.
(P3) There are no self gluings of a Seifert component; that is, for every $A_{\tau,t;\gamma} \in \mathcal{A} \Longrightarrow \tau \neq t$.
(P4) In the graph there appear only 3 types of vertices sf_τ:
  Type 1: The vertex sf_τ is in the boundary of exactly 1 edge and this is trivially marked by $\left(\begin{smallmatrix}0 & 1\\ * & *\end{smallmatrix}\right)$.
  Type 2: The vertex sf_τ is in the boundary of exactly 2 edges where at least one of them is trivially marked.
  Type 3: The vertex sf_τ is in the boundary of exactly 3 trivially marked edges.

The cell decomposition represented by a canonical graph is also called *canonical*.

4.9. Theorem. *Every graph manifold* W *can be represented by a canonical graph.*

Proof. Above we have described the different building stones for the matrices

$$A_{3,2} = \begin{pmatrix}\alpha & \beta\\ & *\end{pmatrix} \quad \text{and} \quad A_{4,5;1} = \begin{pmatrix}a_{4,5;1} & b_{4,5;1}\\ c_{4,5;1} & d_{4,5;1}\end{pmatrix}:$$

$$\begin{tikzcd} \bullet sf_1 \arrow[d] \\ \bullet sf_2 \\ \bullet sf_3 \arrow[u, "A_{3,2}"'] \end{tikzcd}
\qquad
\begin{tikzcd} \bullet sf_3 \arrow[rr] & & \bullet sf_4 \\ & \bullet sf_2 \arrow[ul] \arrow[ur] & \\ & \bullet sf_1 \arrow[u] & \end{tikzcd}
\qquad
\begin{tikzcd} \bullet sf_1 \arrow[r] & \bullet sf_2 \arrow[r] \arrow[d] & \bullet sf_3 \\ & \bullet sf_4 \arrow[d, "A_{4,5;1}"] & \\ & \bullet sf_5 \arrow[d] & \\ \bullet sf_5 \arrow[r] & \bullet sf_7 \arrow[r] & \bullet sf_8 \end{tikzcd}$$

W_r $\qquad$ W_g $\qquad$ W_v

Following 4.5, we construct for each Seifert component $SF_1, \dots, SF_N$ a linear chain of $g + r + 2$ vertices and trivial marked edges connecting them. Next we apply the constructions 4.2–4.7 to the gluings. $\square$

4.10. Definition.

(a) An edge path in the graph *preserves the Seifert fibration* mod 2 if all of its edges are marked by a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod 2, \quad b \in \{0, 1\}.$$

(b) A spanning tree of a graph representing a graph manifold is *maximal with respect to the Seifert fibrations* if the number of its edges marked by a matrix of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ mod 2 is maximal (among all spanning trees); in general it is not unique. In this case the tree is denoted by $\mathcal{B}_{\text{seif}}$.

Given a spanning tree $\mathcal{B}(W^3)$ and an edge $A_{\tau,t} \in \mathcal{G}(W^3) \setminus \mathcal{B}(W^3)$, it follows from standard properties of spanning trees in graphs that there is a uniquely determined simple edge path $v_{\tau,t}$ in $\mathcal{B}(W^3)$ leading from τ to t, and $v_{\tau,t}^{-1} A_{\tau,t}$ is the only simple closed path consisting of $A_{\tau,t}$ and edges from $\mathcal{B}(W^3)$.

4.11. Definition.

$$I = \left\{ v_{\tau,t} \mid \exists A_{\tau,t} \in \mathcal{G}\left(W^3\right) \setminus \mathcal{B}\left(W^3\right) \right\},$$
$$I_{\text{seif}} = \{ v \in I \mid v \text{ preserves the Seifert fibration mod } 2 \}.$$

4.12. Remark. If there is a simple path in $\mathcal{B}(W^3)$ connecting the endpoints of the edge $A_{\tau,t}$ and preserving the Seifert fibration mod 2 then the uniquely determined simple path in $\mathcal{B}_{\text{seif}}(W^3)$ connecting τ and t has also this property.

5. Cohomology ring of a graph manifold

From the canonical graph of $W^3 = ((SF_\tau)_{1 \leqslant \tau \leqslant N}; \mathcal{A})$ we obtain its canonical cell decomposition and the corresponding chain complex; the complexes are special cases of the standard CW-complexes etc. from Section 3.

5.1. Canonical cell decomposition of a graph manifold. The canonical graph of the closed orientable graph manifold $W^3 = ((SF_\tau)_{1\leqslant\tau\leqslant N}; \mathcal{A})$ reflects, by 4.8, the canonical cell decomposition 3.8. The cells are identified with free generators of the cellular chain groups $C_p(W^3)$. The boundaries are described in 3.9. In the following N again denotes the number of Seifert components of W^3, τ runs from 1 to N, t and t' vary over all possibilities according to $\mathcal{A}$. We assume that the gluing cylinder $A_{\tau,t}$ corresponds to

$$\begin{pmatrix} \mu^1_{\tau,t} \\ h^1_\tau \end{pmatrix} = \begin{pmatrix} a_{\tau,t} & b_{\tau,t} \\ c_{\tau,t} & d_{\tau,t} \end{pmatrix} \begin{pmatrix} \nu^1_{\tau,t} \\ h^1_t \end{pmatrix}.$$

dim 0 :	e^0_τ,	$\partial_0 e^0_\tau = 0$	base point in sf_τ;
dim 1 :	h^1_τ,	$\partial_1 h^1_\tau = 0$	normal fiber in sf_τ,
	$\mu^1_{\tau,t}$,	$\partial_1 \mu^1_{\tau,t} = 0$	boundary curve in the
	$\nu^1_{t',\tau}$,	$\partial_1 \nu^1_{t',\tau} = 0$	base Z_τ of sf_τ,
	$\varepsilon^1_{\tau,t}$,	$\partial_1 \varepsilon^1_{\tau,t} = e^0_t - e^0_\tau$	"connecting" segments of $A_{\tau,t}$;
dim 2 :	e^2_τ,	$\partial_2 e^2_\tau = \sum_t \mu^1_{\tau,t} + \sum_{t'} \nu^1_{t',\tau}$	2-cell of Z_τ,
	$\mu^2_{\tau,t}$,	$\partial_2 \mu^2_{\tau,t} = 0$	"departing" torus,
	$\nu^2_{t',\tau}$,	$\partial_2 \nu^2_{\tau,t} = 0$	"arriving" torus in sf_τ;
	$\varepsilon^2_{\tau,t}$,	$\partial_2 \varepsilon^2_{\tau,t} = \mu^1_{\tau,t} - a_{\tau,t}\nu^1_{\tau,t} - b_{\tau,t} h^1_t$,	
	$\eta^2_{\tau,t}$,	$\partial_2 \eta^2_{\tau,t} = h^1_\tau - c_{\tau,t}\nu^1_{\tau,t} - d_{\tau,t} h^1_t$;	
dim 3 :	e^3_τ,	$\partial_3 e^3_\tau = \sum_t \mu^2_{\tau,t} + \sum_{t'} \nu^2_{t',\tau}$	interior of sf_τ,
	$\varepsilon^3_{\tau,t}$,	$\partial_3 \varepsilon^3_{\tau,t} = \mu^2_{\tau,t} - \nu^2_{\tau,t}$	interior of gluing torus $A_{\tau,t}$.

When building the boundaries some cells appear twice with different signs like

$$\partial_3 \varepsilon^3_{\tau,t} = \mu^2_{\tau,t} + \varepsilon^2_{\tau,t} + \eta^2_{\tau,t} - \varepsilon^2_{\tau,t} - \eta^2_{\tau,t} - \nu^2_{\tau,t} = \mu^2_{\tau,t} - \nu^2_{\tau,t}.$$

When we go to cohomology mod 2 we find, copying the method for Seifert manifolds, the generating cochains of $\mathrm{Hom}(C_p(W^3), \mathbb{Z}_2)$ and their coboundaries.

5.2. Cochains and coboundaries.

dim 0 :	$\tilde{e}^0_\tau$,	$\delta_0 \tilde{e}^0_\tau = \sum_t \tilde{\varepsilon}^1_{\tau,t} + \sum_{t'} \tilde{\varepsilon}^1_{t',\tau}$;	
dim 1 :	$\tilde{h}^1_\tau$,	$\delta_0 \tilde{h}^1_\tau = \sum_t \tilde{\eta}^2_{\tau,t} - \sum_{t'} b_{t',\tau} \tilde{\varepsilon}^2_{t',\tau} - \sum_{t'} d_{t',\tau} \tilde{\eta}^2_{t',\tau}$,	$(*)$
	$\tilde{\mu}^1_{\tau,t}$,	$\delta_1 \tilde{\mu}^1_{\tau,t} = \tilde{e}^2_\tau + \tilde{\varepsilon}^2_{\tau,t}$,	
	$\tilde{\nu}^1_{t',\tau}$,	$\delta_1 \tilde{\nu}^1_{t',\tau} = \tilde{e}^2_\tau - c_{t',\tau} \tilde{\eta}^2_{t',\tau} - a_{t',\tau} \tilde{\varepsilon}^2_{t',\tau}$,	$(**)$
	$\tilde{\varepsilon}^1_{\tau,t}$,	$\delta_1 \tilde{\varepsilon}^1_{\tau,t} = 0$;	
dim 2 :	$\tilde{e}^2_\tau$,	$\delta_2 \tilde{e}^2_\tau = 0$,	
	$\tilde{\mu}^2_{\tau,t}$,	$\delta_2 \tilde{\mu}^2_{\tau,t} = \tilde{e}^3_\tau + \tilde{\varepsilon}^3_{\tau,t}$,	
	$\tilde{\nu}^2_{t',\tau}$,	$\delta_2 \tilde{\nu}^2_{t',\tau} = \tilde{e}^3_\tau + \tilde{\varepsilon}^3_{t',\tau}$,	
	$\tilde{\varepsilon}^2_{\tau,t}$,	$\delta_2 \tilde{\varepsilon}^2_{\tau,t} = 0$,	
	$\tilde{\eta}^2_{\tau,t}$,	$\delta_2 \tilde{\eta}^2_{\tau,t} = 0$;	

$$\begin{aligned}\dim 3:\quad &\tilde e^3_\tau, &&\delta_3\tilde e^3_\tau = 0,\\ &\tilde\varepsilon^3_{\tau,t}, &&\delta_3\tilde\varepsilon^3_{\tau,t} = 0.\end{aligned}$$

In Theorem 5.5 we describe the cohomology groups and determine "most of the essential" $\cup$-products for their generators. Non-"essential" are $\cup$-products with the generator of $H^0(W^3,\mathbb{Z}_2)$—which behaves as a unit element—and calculations using the (anti)commutativity of the $\cup$-product. The expression "most" indicates that for some cases the $\cup$-product is not yet determined. The many possible cases of graph manifolds lead to some rather complicated constructions of cocycles. We give a description of them and formulate the main theorem in 5.3 and 5.5. To get a better impression of the cocycles $\Phi^{C_z^=}$ (to be defined) from 5.5 (b) and its construction in 5.3 see Example 5.4.

5.3. Assumptions and constructions. Let

$$W^3 = \left((sf_1,\dots,sf_N);\ \mathcal{A} = \left\{A_{\tau,t} = \begin{pmatrix} a_{\tau,t} & b_{\tau,t}\\ c_{\tau,t} & d_{\tau,t}\end{pmatrix}\right\}\right)$$

be an orientable 3-dimensional graph manifold with a canonical cell decomposition 4.8, and let $\mathcal{G}(W^3)$ be its canonical graph with a fixed spanning tree $\mathcal{B}_{\mathrm{seif}}(W^3)$.

Two vertices t,s of the graph are called *Seifert equivalent* mod 2 if they can be joined by an edge path preserving the Seifert fibrations mod 2. The equivalence classes form m connected subgraphs C_z called *regions* consisting of Seifert equivalent mod 2 vertices and all Seifert fibration preserving edges between them. Such a region C_z corresponds mod 2 to a Seifert manifold; in 2.8 an important indicator has been the number n_z counting the number of exceptional fibers with $\alpha\equiv 0 \bmod 2$. The region C_z is denoted by $C_z^>$ if $n_z>0$ and by $C_z^=$ if $n_z=0$. We order the regions such that $C_1^>,\dots,C_\zeta^>$ and $C_{\zeta+1}^=,\dots,C_m^=$ for an appropriate ζ.

To form a cocycle from the cochains marked $(*)$ and $(**)$ in 5.2, using the coboundary formulae there, some of the 1-cochains of the form $\tilde h^1_\tau$ and $\tilde\nu^1_{t',\tau}$ have to be joined and this can be found out from the graph for $\mathcal{A}$. The η-edges in $C_z^>$ form maximal simple paths such that the gluing along them preserves the Seifert structure mod 2, $\tilde\nu_{r_k,r_k'}$ denotes the first edge of such a path. (This corresponds to 5.5 (b)($\bullet$).) Because of the coboundary in 5.2 marked by $(**)$ one has to put together more terms since $a_{t',\tau}\equiv 1 \bmod 2$, and this defines a *system* $L_{C_z^=}$ *of edges* appearing in the two expressions from 5.5 (b) marked by ($\bullet\bullet$).

When looking at the coboundaries of the $\tilde h^1_\tau$, $\tilde\mu^1_{\tau,t}$ etc. it becomes clear that for building cocycles one has to use appropriate sums. The coboundary of the $\tilde\nu^1_{t',\tau}$ shows that the cases $a_{t',\tau}\equiv 0 \bmod 2$ and $a_{t',\tau}\equiv 1 \bmod 2$ and therefore the regions $C_z^>$ and $C_z^=$ behave differently. We obtain two types of cohomology classes denoted by Ψ and Φ.

For the existence of a class Φ mainly formed by the sums of the $\tilde h^1_\tau$ belonging to a region $C_z^=$, it is necessary to eliminate the "error"-terms of $\delta\sum\{\tilde h^1_\tau \mid \tau\colon sf_\tau\subset C_z^=\}$ which are cocells of the form $\tilde\varepsilon^2_{\tau,t'}$ and $\tilde\eta^2_{\tau,t'}$. They are obtained from mod 2 non-trivial gluings. Such an "error"-term can only be eliminated by additional "$\mu\nu$"-chains, that is, by sums of expressions $\tilde\mu^1_{\tau,t}+\tilde\nu^1_{\tau,t}$, which connect adequate gluing cylinders and are uniquely determined by the spanning tree $\mathcal{B}_{\mathrm{seif}}(W^3)$. So we need for every "error"-gluing another "error"-gluing which can be reached along a mod 2 Seifert fibration preserving path. (Since the regions C_z

are maximal with respect to mod 2 Seifert equivalence, and thus the problem can be handled for each region C_z as a unit.) Moreover all "error"-terms in a $C_z^>$-region can be joined to an edge representing an exceptional fiber of type $(\alpha,\beta) \equiv (0,1)$ mod 2 and this makes redundant the consideration of an even number of "error"-gluings to $C_z^>$.

For the other regions $C^=_{\zeta+1},\dots,C^=_m$ we look to the targets of the edges and build some matrix P. (Compare Example 5.4). The rows and columns correspond to the regions $C^=_z$, $\zeta+1 \leqslant z \leqslant m$. We denote the diagonal elements by ξ_z, the element at the position (z_1,z_2) by $\xi_{z_1\to z_2}$ if $z_1 \neq z_2$; the coefficients are considered as elements from the field $\mathbb{Z}_2$. Starting with the null-matrix we change the coefficients when looking at the gluings according to the following rules; here $\xi_z \nearrow$ means that the coefficient ξ_z has to be increased by 1 etc., and a gluing matrix A is replaced mod 2 by a matrix with coefficients 0 and 1. Gluings with the unit matrix have already been used for the construction of the components C_z. The remaining cases are:

matrix	gluing from ... to ...	coefficient change
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$	C_z	$\xi_z \nearrow$
$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$C_z \to C_{z'}$	$\xi_z \nearrow, \xi_{z\to z'} \nearrow$
$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	$C_z \to C_{z'}$	$\xi_z \nearrow, \xi_{z\to z'} \nearrow$
$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$C_z \to C_{z'}$	$\xi_{z\to z'} \nearrow$
$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$C_z \to C_{z'}$	$\xi_{z\to z'} \nearrow$
$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$C_{z'} \to C_z$	$\xi_z \nearrow, \xi_{z\to z'} \nearrow$

The *dimension ϱ of the kernel* of the linear mapping corresponding to the matrix P over $\mathbb{Z}_2$ indicates how many independ cocycles are obtained between the $C^=_z$; assume that they are obtained from $C^=_{\zeta+1},\dots,C^=_{\zeta+\varrho}$ by attaching the remaining ϱ regions.

5.4. Example. Consider the graph manifold

$$W^3 = \Bigg(\big(C_1^= = sf_1,\ C_2^= = sf_2\big); \\ \mathcal{A} = \left\{ A_1 = A_{1,2;1} = \begin{pmatrix}1&0\\1&1\end{pmatrix},\ A_2 = A_{1,2;2} = \begin{pmatrix}1&0\\1&1\end{pmatrix} \right\} \Bigg).$$

It has the graph $\underset{sf_1}{\bullet} \overset{A_1}{\underset{A_2}{\rightrightarrows}} \underset{sf_2}{\bullet}$. According to 5.1 we obtain the following cells and boundaries:

$$\begin{array}{lll} \dim 0: & e_1^0, e_2^0, & \partial_0 e_j^0 = 0; \\ \dim 1: & h_1^1, h_2^1, & \partial_1 h_j^1 = 0, \\ & \mu^1_{1,2;1^1}, \mu^1_{1,2;2}, & \partial_1 \mu^1_{1,2;j} = 0, \end{array}$$

$$\begin{array}{lll}
 & \nu^1_{1,2;1}, \nu^1_{1,2;2}, & \partial_1 \nu^1_{1,2;j} = 0, \\
 & \varepsilon^1_{1,2;1}, \varepsilon^1_{1,2;2}, & \partial_1 \varepsilon^1_{1,2;j} = e^0_2 - e^0_1; \\
\dim 2: & e^2_1, & \partial_2 e^2_1 = \mu^1_{1,2;1} + \mu^1_{1,2;2}, \\
 & e^2_2, & \partial_2 e^2_1 = \mu^1_{1,2;1} + \mu^1_{1,2;2}, \\
 & \mu^2_{1,2;1}, \mu^2_{1,2;2}, & \partial_2 \mu^2_{1,2;j} = 0, \\
 & \nu^2_{1,2;1}, \nu^2_{1,2;2}, & \partial_2 \nu^2_{1,2;j} = 0, \\
 & \varepsilon^2_{1,2;1}, \varepsilon^2_{1,2;2}, & \partial_2 \varepsilon^2_{1,2;j} = \mu^1_{1,2;j} - \nu^1_{1,2;j}, \\
 & \eta^2_{1,2;1}, \eta^2_{1,2;2}, & \partial_2 \eta^2_{1,2;j} = h^1_1 - \nu^1_{1,2;j} - h^1_2; \\
\dim 3: & e^3_1, & \partial_3 e^3_1 = \mu^2_{1,2;1} + \mu^2_{1,2;2}, \\
 & e^3_2, & \partial_3 e^3_1 = \nu^2_{1,2;1} + \nu^2_{1,2;2}, \\
 & \varepsilon^3_{1,2;1}, \varepsilon^3_{1,2;2}, & \partial_3 \varepsilon^3_{1,2;j} = \mu^2_{1,2;j} - \nu^2_{1,2;j}.
\end{array}$$

According to 5.2 we obtain the corresponding cocells:

$$\begin{array}{lll}
\dim 0: & \tilde{e}^0_1, \tilde{e}^0_1, & \delta_0 \tilde{e}^0_j = \tilde{\varepsilon}^1_{1,2;1} + \tilde{\varepsilon}^1_{1,2;2}; \\
\dim 1: & \tilde{h}^1_1, \tilde{h}^1_2, & \delta_1 \tilde{h}^1_j = \tilde{\eta}^2_{1,2;1} + \tilde{\eta}^2_{1,2;2}, \\
 & \tilde{\mu}^1_{1,2;1}, \tilde{\mu}^1_{1,2;1}, & \delta_1 \tilde{\mu}^1_{1,2;j} = \tilde{e}^2_1 + \tilde{\varepsilon}^2_{1,2;j}, \\
 & \tilde{\nu}^1_{1,2;1}, \tilde{\nu}^1_{1,2;1}, & \delta_1 \tilde{\nu}^1_{1,2;j} = \tilde{e}^2_2 + \tilde{\eta}^2_{1,2;j} + \tilde{\varepsilon}^2_{1,2;j}, \\
 & \tilde{\varepsilon}^1_{1,2;1}, \tilde{\varepsilon}^1_{1,2;1}, & \delta_1 \tilde{\varepsilon}^1_{1,2;j} = 0; \\
\dim 2: & \tilde{e}^2_1, \tilde{e}^2_2, & \delta_2 \tilde{e}^2_j = 0, \\
 & \tilde{\mu}^2_{1,2;1}, \tilde{\mu}^2_{1,2;2}, & \delta_2 \tilde{\mu}^2_{1,2;j} = \tilde{e}^3_1 + \tilde{\varepsilon}^3_{1,2;j}, \\
 & \tilde{\nu}^2_{1,2;1}, \tilde{\nu}^2_{1,2;2}, & \delta_2 \tilde{\nu}^2_{1,2;j} = \tilde{e}^3_2 + \tilde{\varepsilon}^3_{1,2;j}, \\
 & \tilde{\varepsilon}^2_{1,2;1}, \tilde{\varepsilon}^2_{1,2;2}, & \delta_2 \tilde{\varepsilon}^2_{1,2;j} = 0, \\
 & \tilde{\eta}^2_{1,2;1}, \tilde{\eta}^2_{1,2;2}, & \delta_2 \tilde{\eta}^2_{1,2;j} = 0; \\
\dim 3: & \tilde{e}^3_1, \tilde{e}^3_2, & \delta_3 \tilde{e}^3_j = 0, \\
 & \tilde{\varepsilon}^3_{1,2;1}, \tilde{\varepsilon}^3_{1,2;2}, & \delta_3 \tilde{\varepsilon}^3_{1,2;j} = 0.
\end{array}$$

To get the matrix P from 5.3 we use the following table where the subscript of a coefficient indicates the matrix the "1" comes from; P is obtained from the table by taking the entries mod 2.

	$C_1^=$	$C_2^=$
$C_1^=$	$1_1 + 1_2$	$1_1 + 1_2$
$C_2^=$	$1_1 + 1_2$	$1_1 + 1_2$

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{rank} P = 0, \ \varrho = 4 - 2 = 2.$$

Hence, we obtain the following independent generators of $H^p(W^3; \mathbb{Z}_2)$; our main interest in this example lies in the two cases marked with a (!!). (Here $\nexists$ indicates that the corresponding cohomology classes do not exist, while ■ expresses that the answer is not known.)

$$\begin{aligned}
p=0:\quad & 1=\{\tilde e_1^0+\tilde e_2^0\},\\
p=1:\quad & \Psi' \nexists,\\
& \Psi''=\{\tilde\varepsilon^1_{1,2;1}\}=\{\tilde\varepsilon^1_{1,2;2}\},\\
& \Phi^{C_1^=}=\{\tilde h_1^1+\tilde\mu^1_{1,2;1}+\tilde\nu^1_{1,2;1}+\tilde\mu^1_{1,2;2}+\tilde\nu^1_{1,2;2}\}, \quad (!!)\\
& \Phi^{C_2^=}=\{\tilde h_2^1+\tilde\mu^1_{1,2;1}+\tilde\nu^1_{1,2;1}+\tilde\mu^1_{1,2;2}+\tilde\nu^1_{1,2;2}\}, \quad (!!)\\
p=2:\quad & \Gamma'=\{\tilde\mu^2_{1,2;1}+\tilde\nu^2_{1,2;1}+\tilde\mu^2_{1,2;2}+\tilde\nu^2_{1,2;2}\},\\
& \Gamma'' \nexists,\\
& \Lambda^{C_1^=}=\{\tilde e_1^2\}=\{\tilde\varepsilon^2_{1,2;1}\}=\{\tilde\varepsilon^2_{1,2;2}\},\\
& \Lambda^{C_2^=}=\{\tilde e_2^2\}=\{\tilde\eta^2_{1,2;1}\}=\{\tilde\eta^2_{1,2;2}\};\\
p=3:\quad & \Theta=\{\tilde e_1^3\}=\{\tilde e_2^3\}=\{\tilde\varepsilon^3_{1,2;1}\}=\{\tilde\varepsilon^3_{1,2;2}\}.
\end{aligned}$$

From 5.5 we see that

$$\begin{array}{llll}
\Psi''\cup\Psi''=0, & \Psi''\cup\Phi^{C_j^=}=0, & \Phi^{C_i^=}\cup\Phi^{C_j^=}=\blacksquare, & \\
\Psi''\cup\Gamma'=\Theta, & \Psi''\cup\Lambda^{C_i^=}=0, & \Phi^{C_i^=}\cup\Gamma'=\blacksquare, & \Phi^{C_i^=}\cup\Lambda^{C_i^=}=\Theta.
\end{array}$$

Unfortunately, there are still cases where the $\cup$-product is not known.

Next we formulate the main theorem. (For $\nexists$ and ■ see above.)

5.5. Theorem. *Assume that the assumptions etc. from* 5.3 *are fulfilled.*

(a) *The cohomology groups are*

$$\begin{aligned}
&H^0(W^3;\mathbb{Z}_2)\cong H^3(W^3;\mathbb{Z}_2)\cong\mathbb{Z}_2,\\
&H^1(W^3;\mathbb{Z}_2)\cong H^2(W^3;\mathbb{Z}_2)\cong\mathbb{Z}_2^\kappa;
\end{aligned}$$

here $\kappa=|I|+|I_{\mathrm{seif}}|+\sum_{k=1}^{\zeta}(n_k-1)+\varrho$.

(b) *Independent* ($\mathbb{Z}_2$-)*generators are the following.*

$$\begin{aligned}
\dim 0:\quad & 1=\{\textstyle\sum_{\tau=1}^N\tilde e_\tau^0\},\\
\dim 1:\quad & \Psi_i'=\{\textstyle\sum_{(\tau,t)\in\ell_i}(\tilde\mu^1_{\tau,t}+\tilde\nu^1_{\tau,t})\} \quad \textit{for } \ell_i\in I_{\mathrm{seif}},\\
& \Psi_j''=\{\tilde\varepsilon^1_{\tau,t}\} \quad \textit{with } (\tau,t)\in\ell_j \quad \textit{for } \ell_j\in I,\\
& \Phi_k^{C_z^>}=\{\tilde\nu_{r_k,r_k'}+\textstyle\sum_{(\tau,t)\in v_{r_k',n_z}}(\tilde\mu^1_{\tau,t}+\tilde\nu^1_{\tau,t})+\tilde\nu^1_{n_z,n_z'}\}\\
& \qquad \textit{for } 1\leqslant k\leqslant n_z-1,\ 1\leqslant z\leqslant\zeta, \qquad (\bullet)\\
& \Phi^{C_z^=}=\{\textstyle\sum_{v_{r,s}\in L_{C_z^=}}(\tilde\nu^1_{r,r'}+\sum_{(t,t')\in v_{r,s}}(\tilde\mu^1_{t,t'}+\tilde\nu^1_{t,t'})+\tilde\nu^1_{s,s'})\\
& \qquad +\textstyle\sum_{\tau\in C_z^=}h_\tau^1\}, \quad \zeta\leqslant z\leqslant\zeta+\varrho, \qquad (\bullet\bullet)\\
\dim 2:\quad & \Gamma_j'=\{\textstyle\sum_{(\tau,t)\in\ell_j}(\tilde\mu^2_{\tau,t}+\tilde\nu^2_{\tau,t})\} \quad \textit{for } \ell\in I,\\
& \Gamma_i''=\{\tilde\eta^2_{\tau,t}\} \quad \textit{with } (\tau,t)\in\ell_i \quad \textit{for } \ell_i\in I_{\mathrm{seif}},\\
& \Lambda_k^{C_z^>}=\{\tilde\eta^2_{\tau,t}\} \quad \textit{with } (\tau,t)\in v_{r_k,n_z} \quad \textit{for } 1\leqslant k\leqslant n_z-1,
\end{aligned}$$

$$\Lambda^{C_z^=} = \{\tilde{e}_\tau^2\} = \{\tilde{\varepsilon}_{\tau,t}^2\} = \{\tilde{\varepsilon}_{t',\tau}^2\} = \{\tilde{\eta}_{t',t''}^2\} \quad \textit{with } \tau \in C_z^=,$$
$$(t', t'') \in \upsilon \in L_{C_z^=} \quad \textit{if there is a } C_z^=; \qquad (\bullet\bullet)$$
$$\dim 3: \quad \Theta = \{\tilde{e}_\tau^3\} = \{\tilde{\varepsilon}_{\tau,t}^3\} = \{\tilde{\varepsilon}_{t',\tau}^3\}, \ \textit{for } 1 \leqslant \tau \leqslant N, t \in N(\tau).$$

(c) *Consider* $1 \leqslant i \leqslant |I|$, $1 \leqslant j \leqslant |I_{\mathrm{seif}}|$. *By assumption*, n_z *is the highest index of an exceptional fiber of type* $(\alpha, \beta) \equiv (0, 1) \bmod 2$ *in* C_z. *Moreover* $\Lambda_{n_z}^{C_z^>} = \sum_{k=1}^{n_z-1} \Lambda_{n_k}^{C_z^>}$. *Then the non-trivial* $\cup$*-products of* W^3 *are as follows.*

$$\Psi_i'' \cup \Psi_j' = \begin{cases} \delta_{ij} \Lambda^{C_z^=} & \textit{if } \Phi^{C_z^=} \textit{ exists,} \\ 0 & \textit{otherwise,} \end{cases}$$
$$\Phi_k^{C_z^>} \cup \Phi_l^{C_z^>} = \delta_{kl} \binom{\alpha_l}{2} \Lambda_l^{C_z^>} + \binom{\alpha_{n_z}}{2} \Lambda_{n_z}^{C_z^>},$$
$$\Phi_k^{C_z^>} \cup \Phi^{C_{z'}^=} = \binom{\alpha_{n_z}}{2} \Lambda_{n_z}^{C_z^>}, \quad \textit{if } \Phi^{C_z^=} | C_z^> \neq 0,$$
$$\Phi^{C_z^=} \cup \Phi^{C_{z'}^=} = \blacksquare,$$
$$\Psi_j'' \cup \Gamma_i' = \delta_{ij} \Theta,$$
$$\Psi_j' \cup \Gamma_i'' = \delta_{ij} \Theta,$$
$$\Phi^{C_z^=} \cup \Gamma_i' = \blacksquare,$$
$$\Phi_k^{C_z^>} \cup \Lambda_l^{C_z^>} = \delta_{kl} \Theta, \quad \textit{for } 1 \leqslant k, l \leqslant n_z,$$
$$\Phi^{C_z^=} \cup \Lambda^{C_z^=} = \Theta.$$

Remarks to the proof. Since there are many types of graph manifolds with different invariants we have to expect that there are many cases for the cohomology groups, but the calculation reduces to the solution of systems of linear equations. So the claims (a) and (b) on the groups and its generators can be proved by standard arguments.

The calculation of $\cup$-products is done using suitable decompositions of W^3 and reducing the calculation to that of Seifert manifolds applying 1.8, 1.12. Unfortunately, different types of decompositions $W^3 = A \cup B$ with $A = \overline{W^3 \setminus B}$ are needed for different cohomology classes.

The first type is based on the fact that the classes Ψ_i' and Γ_i'' vanish outside the Seifert component they are from.

The second type deals with Φ_k and Λ_k which belong to exceptional fibers of type $(\alpha, \beta) \equiv (0, 1) \bmod 2$ in Seifert components or in regions. They are represented by cochains which admit non-zero values only on paths which preserve the Seifert fibration. Following [1], there are constructed special spaces with a quite large structure containing at least the exceptional fibers of the above type appearing in the region.

The third kind to construct a Seifert manifold is based on the fact that the classes Λ_k, Ψ_i'' and Γ_i'' are represented by single "cocells". The complex B is represented by a single edge of type $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ such that the considered classes admit non-zero values only in B; it follows that the classes vanish on A. The space B has a trivial Seifert fibration and can be extended by a simple A' realizing the trivial identification of the two boundary tori of B. For this step see the following Example 5.6.

5.6. Example: Calculating some ∪-products. We consider the following classes represented by single "cocells" $\Psi_i'' = \{\tilde{\varepsilon}^1_{\tau,t}\}$ and $\Gamma_i'' = \{\tilde{\eta}^2_{\tau,t}\}$, respectively, located on a trivial marked edge. As part of a path which contains mod 2 Seifert fibrations preserving edges except one non-preserving 1-cell we obtain

$$\cdots \longrightarrow \bullet_{sf_\tau} \longrightarrow \bullet_{sf_t} \xrightarrow{\left(\begin{smallmatrix} * & * \\ 1 & * \end{smallmatrix}\right)} \bullet \longrightarrow \cdots.$$

We apply 1.8 and take as B the space consisting of sf_τ and sf_t, each with one boundary torus, and the trivial connecting gluing $A_{\tau,t} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It can be considered as a subcomplex of W^3. As A we take the closure of the complement. On the other side we extend B by $A' = \left(sf_x;\ \left\{A_{x,\tau} = A_{x,t} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}\right)$. Hence the resulting space is the following Seifert manifold M (together with its graph):

$$M = \left((sf_x, sf_\tau, sf_t);\ \left\{A_{x,\tau} = A_{x,t} = A_{t,\tau} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right)$$

$$\begin{tikzcd} & \bullet_{sf_\tau} \arrow[d] \\ \bullet_{sf_x} \arrow[ur] \arrow[r] & \bullet_{sf_t}. \end{tikzcd}$$

This M admits—using the notation from 2.9—$n = 0$ and $\beta \equiv 0 \bmod 2$. We mark the cohomology classes by the spaces they are from, but we will not do so for the cochains and the same symbol is used for "corresponding" cochains for different spaces. Now the conditions (1′) and (2) of 1.8 have to be proved for the dimensions 1 and 2. The space B has a torus as deformation retract, so the cohomology classes are known:

$$\begin{aligned}
&H^0(B) \cong \mathbb{Z}_2 && \text{generated by } 1^B = \{\tilde{e}^0_\tau + \tilde{e}^0_t\},\\
&H^1(B) \cong \mathbb{Z}_2^2 && \text{generated by } {\Psi'}_1^B = \{\tilde{\mu}^1_{\tau,t} + \tilde{\nu}^1_{\tau,t}\},\ \Phi^B = \{\tilde{h}^1_\tau\} + \{\tilde{h}^1_t\},\\
&H^2(B) \cong \mathbb{Z}_2 && \text{generated by } {\Gamma'}_1^B = \{\tilde{\mu}^2_{\tau,t} + \tilde{\nu}^2_{\tau,t}\},\\
&H^3(B) = 0.
\end{aligned}$$

The cohomology groups of M are given by:

$$\begin{aligned}
&H^0(M) \cong \mathbb{Z}_2 && \text{generated by } 1^M = \{\tilde{e}^0_\tau + \tilde{e}^0_t + \tilde{e}^0_x\},\\
&H^1(M) \cong \mathbb{Z}_2^3 && \text{generated by}\\
& && \quad {\Psi'}_1^M = \{\tilde{\mu}^1_{\tau,t} + \tilde{\nu}^1_{\tau,t} + \tilde{\mu}^1_{x,t} + \tilde{\nu}^1_{x,t} + \tilde{\mu}^1_{x,\tau} + \tilde{\nu}^1_{x,\tau}\},\\
& && \quad \Phi^M = \{\tilde{h}^1_\tau + \tilde{h}^1_t + \tilde{h}^1_x\},\\
& && \quad {\Psi'''}_1^M = \{\tilde{\varepsilon}^1_{\tau,t}\} = \{\tilde{\varepsilon}^1_{x,\tau}\} = \{\tilde{\varepsilon}^1_{x,t}\};\\
&H^2(M) \cong \mathbb{Z}_2^3 && \text{generated by}\\
& && \quad {\Gamma'}_1^M = \{\tilde{\mu}^2_{\tau,t} + \tilde{\nu}^2_{\tau,t} + \tilde{\mu}^2_{x,t} + \tilde{\nu}^2_{x,t} + \tilde{\mu}^2_{x,\tau} + \tilde{\nu}^2_{x,\tau}\},\\
& && \quad {\Gamma''}_1^M = \{\tilde{\eta}^2_{\tau,t}\} = \{\tilde{\eta}^2_{x,\tau}\} = \{\tilde{\eta}^2_{x,t}\},\\
& && \quad \Lambda^M = \{\tilde{e}^2_\tau\} = \{\tilde{\varepsilon}^2_{\tau,t}\} = \{\tilde{e}^2_t\} = \{\tilde{\varepsilon}^2_{x,t}\} = \{\tilde{e}^2_x\} = \{\tilde{\varepsilon}^2_{x,\tau}\};\\
&H^3(M) \cong \mathbb{Z}_2 && \text{generated by}\\
& && \quad \Theta^M = \{\tilde{e}^3_\tau\} = \{\tilde{\varepsilon}^3_{\tau,t}\} = \{\tilde{e}^3_t\} = \{\tilde{\varepsilon}^3_{x,t}\} = \{\tilde{e}^3_x\} = \{\tilde{\varepsilon}^3_{x,\tau}\}.
\end{aligned}$$

For the dimensions 1 and 2 the surjectivity of i'^* onto Im i^* can explicitly be seen by the non-trivial part of Im i^* where the index ℓ indicates the classes taking non-zero values on the chosen edge. Now

$$i^*\left(\Psi_\ell'^{\,W}\right) = \Psi_1'^{\,B} = i'^*\left(\Psi_1'^{\,M}\right), \qquad i^*\left(\Phi^{C_z^{=W}}\right) = \Phi^B = i'^*\left(\Phi^M\right)$$

if $\Phi^{C_z^W}$ exists, and $i^*(\Gamma_\ell'^{\,W}) = \Gamma_1'^{\,B} = i'^*(\Gamma_1'^{\,M})$.

Condition $(1')$ follows directly from the cohomology classes of

$$H^2\left(M, A'\right) = \left\langle \Lambda^{(M,A')} = \left\{\tilde{e}_\tau^2\right\} = \left\{\tilde{\varepsilon}_{\tau,t}^2\right\} = \left\{\tilde{e}_\tau^2\right\}\right\rangle \cong \mathbb{Z}_2, \quad \text{and}$$
$$H^3\left(M, A'\right) = \left\langle \Theta^{(M,A')} = \left\{\tilde{e}_\tau^3\right\} = \left\{\tilde{\varepsilon}_{\tau,t}^3\right\} = \left\{\tilde{e}_\tau^3\right\}\right\rangle \cong \mathbb{Z}_2,$$

because they are mapped by j^* onto classes in $H^{p+q}(M)$ having the same name and are represented by the same cocells.

Next we will calculate some $\cup$-products applying the formula from 1.8. Using

$$\begin{aligned}
&\Psi_i''^{\,M} = \left[j'^* \circ \left(\mathrm{exc}'\right)^{-1} \circ \mathrm{exc} \circ (j^*)^{-1}\right]\left(\Psi_i''^{\,W}\right),\\
&\Psi_j'^{\,M} = \left[(i^*)^{-1} \circ i^*\right]\left(\Psi_j'^{\,W}\right),\\
&\Psi_i''^{\,M} \cup \Psi_j'^{\,M} = \begin{cases} 0 & \text{for } i \neq j,\\ \Lambda^M & \text{for } i = j,\end{cases}\\
&\Gamma_j'^{\,M} = \left[j'^* \circ \left(\mathrm{exc}'\right)^{-1} \circ \mathrm{exc} \circ (j^*)^{-1}\right]\left(\Gamma_j'^{\,W}\right),\\
&\Psi_i''^{\,M} \cup \Gamma_j'^{\,M} = \delta_{ij}\Theta^M,
\end{aligned}$$

we obtain from 1.8 (putting $\chi = j^* \circ (\mathrm{exc})^{-1} \circ \mathrm{exc}' \circ (j^*)^{-1}$):

$$\Psi_i''^{\,W} \cup \Psi_j'^{\,W} = \chi\left(\Phi_i''^{\,M} \cup \Psi_j'^{\,M}\right) = \begin{cases} \chi\left(\Psi_i''^{\,M} \cup 0\right) = 0, & i \neq j,\\ \chi\left(\Lambda^M\right) = \Lambda^{C_z^{=W}}, & i = j;\end{cases}$$
$$\Psi_i''^{\,W} \cup \Gamma_j'^{\,W} = \chi\left(\Psi_i''^{\,M} \cup \Gamma_j'^{\,M}\right) = \begin{cases} \chi\left(\Psi_i''^{\,M} \cup 0\right) = 0, & i \neq j,\\ \chi\left(\Theta^M\right) = \Theta^W, & i = j;\end{cases}$$

here we put $\Lambda^{C_z^{=W}} = 0$ if there is no class $\Psi_i''^{\,W}$ coming from the region $C_z^{=W}$. This verifies some of the $\cup$-product formulae from 5.5.

6. Kinks in the cohomology of graph manifolds

The homotopy classification of Lorentz metrics plays a role in the relativity theory of a space–time manifold $M^3 \times \mathbb{R}$ where M^3 is an orientable closed connected 3-manifold. The set of homotopy classes of Lorentz metrics is homotopy equivalent to $[M^3, SO(3)] \cong [M, \mathbb{R}P^3]$, and different homotopy classes are called *kinks*. From the universal cover $S^3 \to \mathbb{RP}^3$ and the inclusion $\mathbb{RP}^3 \to K(\mathbb{Z}_2, 1)$ we obtain the following short exact sequence of groups

$$0 \to \mathbb{Z} \cong \left[M^3, S^3\right] \to \left[M^3, \mathbb{RP}^3\right] \to H^1(M; \mathbb{Z}_2) \to 0. \tag{$*$}$$

By [11], $[M^3, \mathbb{RP}^3]$ is an abelian group and depends only on $H^1(M^3; \mathbb{Z}_2)$ and the condition whether the exact sequence $(*)$ splits or not.

6.1. Definition and notation. Let $H^1(M^3; \mathbb{Z}_2) \cong \mathbb{Z}_2^m$. If $(*)$ does not split then $[M^3, \mathbb{RP}^3] = \mathbb{Z} \oplus \mathbb{Z}_2^{m-1}$, else $[M^3, \mathbb{RP}^3] = \mathbb{Z} \oplus \mathbb{Z}_2^m$. In the first case M^3 *is of type* 1: $\operatorname{type}(M^3) = 1$, in the second case $\operatorname{type}(M^3) = 2$. By [11], $\operatorname{type}(M^3) = 1$ iff there exists a cohomology class $\Omega \in H^1(M^3; \mathbb{Z}_2)$ such that $\Omega \cup \Omega \cup \Omega \neq 0$. It suffices to determine the type of irreducible 3-manifolds since, by [11]

$$\operatorname{type}\bigl(M_1^3 \# M_2^3\bigr) = \min\bigl\{\operatorname{type}\bigl(M_1^3\bigr), \operatorname{type}\bigl(M_2^3\bigr)\bigr\}.$$

From 5.5 we obtain some results about the existence of kinks in graph manifolds.

6.2. Proposition. *Let W^3 be a graph manifold in the canonical presentation* 4.8. *Then the following claims are equivalent.*

(a) *There is a class $\Phi_k^{C_z^>} \in H^1(W^3)$ with the property*

$$\Phi_k^{C_z^>} \cup \Phi_k^{C_z^>} \cup \Phi_k^{C_z^>} = \Theta.$$

(b) W^3 *contains a region $C_z^>$ which admits at least two exceptional fibers of the form* $(\alpha, \beta) \equiv (0, 1) \bmod 2$ *and for one of them* (*with index k*)

$$\left[\binom{\alpha_k}{2} + \binom{\alpha_{n_z}}{2}\right] \equiv 1 \bmod 2.$$

Proof. From the list of $\cup$-products we obtain

$$\begin{aligned}\Phi_k^{C_z^>} \cup \bigl(\Phi_k^{C_z^>} \cup \Phi_k^{C_z^>}\bigr) &= \binom{\alpha_k}{2}\Phi_k^{C_z^>} \cup \Lambda_k^{C_z^>} + \binom{\alpha_{n_z}}{2}\Phi_k^{C_z^>} \cup \Lambda_{n_z}^{C_z^>} \\ &= \left[\binom{\alpha_k}{2} + \binom{\alpha_{n_z}}{2}\right] \cdot \Theta.\end{aligned}$$

For $\operatorname{type}(W^3) = 1$, there is the necessary condition that there is a $C_z^>$-region which has at least two exceptional fibers with $(\alpha, \beta) \equiv (0, 1) \bmod 2$. A sufficiency condition is the existence of an exceptional fiber $(\alpha_k, \beta_k) \equiv (0, 1) \bmod 2$ with $\bigl[\binom{\alpha_k}{2} + \binom{\alpha_{n_z}}{2}\bigr] \equiv 1 \bmod 2$. $\square$

6.3. Corollary.

(a) $\operatorname{type}(W^3) = 1$ *if one of the conditions in* 6.2 *is fulfilled.*
(b) *If W^3 admits no $C_z^=$ and if every $C_z^>$ has at most one exceptional fiber of the form* $(\alpha, \beta) \equiv (0, 1) \bmod 2$ *then* $\operatorname{type}(W^3) = 2$.

A complete answer is not yet obtained since the $\cup$-products $\Phi^{C_z^=} \cup \Phi^{C_{z'}^=}$ and $\Phi^{C_z^=} \cup \Gamma_i'$ are not determined.

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